3 (Sem–1/CBCS) MAT HC 2

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : $1 \times 10 = 10$

   (a) Find the polar representation of $z = 2i$.

   (b) If $x = 0$ and $y > 0$, then what is the value of t^* ?

   (c) Write the negation of the statement 'For any integer n, $n^2 > n$' in plain English then formulate the negation using set of context and quantifier.

(d) Disapprove the statement using counter example :

"For any $x, y \in \mathbb{R}$, $x^2 = y^2$ implies $x = y$."

(e) Suppose f is a constant function from X to Y. The inverse image of a subset of Y cannot be

(i) an empty set

(ii) the whole set X

(iii) a non-empty proper subset of X

(Choose the correct option)

(f) Let $X = Y = \mathbb{R}$. Let $A \subseteq X, B \subseteq Y$. Draw the picture for $A \times B$ where $A = [-1, 1]$ and $B = [2, 3]$.

(g) Suppose a system of linear equations in echelon form has a 3×5 augmented matrix whose fifth column is a pivot column.

Is the system consistent? Justify.

(h) If a set $S = \{\bar{v}_1, \bar{v}_2, ..., \bar{v}_p\}$ in $\mathbb{R}^n$ contains the $\vec{O}$ vector, is the set linearly independent? Justify.

(i) If $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\bar{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, compute $(A\bar{x})^T$.

(j) What is the determinant of an $n \times n$ elementary matrix E that has been scaled by 7.

2. Answer the following questions : 2×5=10

(a) If $z = -2\sqrt{3} - 2i$, find the polar radius and polar argument of z.

(b) Is the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = |x - 2|$ one-one and onto? Explain.

(c) Let universal set be $\mathbb{R}$ and index set be $\mathbb{N}$. For a natural number n, $J_n = \left(0, \frac{1}{n}\right)$.

Identify with justification $\bigcap_{n \in N} J_n$.

(d) Show that T is a linear transformation by finding a matrix that implements the mapping

$$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$$

(e) A is a 2×4 matrix with two pivot positions. Answer the following with justification :

(i) Does $A\vec{x} = \vec{0}$ have a non-trivial solution ?

(ii) Does $A\vec{x} = \vec{b}$ have at least one solution for every $\vec{b}$?

3. Answer ***any four*** questions from the following : 5×4=20

(a) Find the polar representation of the complex number

$z = 1 - \cos a + i \sin a \quad a \in [0, 2\pi)$. 5

(b) Let A and B be subsets of an universal set U. Prove —

(i) $(A \cap B)^C = A^C \cup B^C$

(ii) $(A \cup B)^C = A^C \cap B^C$ 5

(c) Define bijection.

Let $f : \mathbb{N} \to \mathbb{N}$ be $f(m) = m - 1$, if m is even $f(m) = m + 1$, if m is odd. Show f is a bijection and $f^{-1} = f$. 1+4=5

(d) For vectors $\bar{v}_1, \bar{v}_2, ..., \bar{v}_p \in \mathbb{R}^n$ define span $\{\bar{v}_1, \bar{v}_2, ..., \bar{v}_p\}$ construct a 3×3 matrix A with non-zero elements and a vector $\vec{b}$ on $\mathbb{R}^3$ such that $\vec{b}$ is not in the set spanned by the columns of A. 2+3=5

(e) Alka-Seltzer contains sodium bi-carbonate $(NaHCO_3)$ and citric acid $(H_3C_6H_5O_7)$. When a tablet is dissolved in water the following reaction produces sodium citrate, water and carbon dioxide :

$$NaHCO_3 + H_3C_6H_5O_7 \to Na_3C_6H_5O_7 + H_2O + CO_2$$

Balance the chemical equation using vector equation approach. 5

(f) Prove that an $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}. 5

4. Answer ***any four*** from the following : $10\times4=40$

(a) (i) Find the cube roots of the number $z = 1 + i$ and represent them in the complex plane. 5

(ii) Find the number of ordered pairs (a, b) of real numbers such that $(a+ib)^{2002} = a - ib$. 2

(iii) If x, y, z be real numbers such that $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$, prove that $\sin 2x + \sin 2y + \sin 2z = 0$ and $\cos 2x + \cos 2y + \cos 2z = 0$. 3

(b) (i) Solve the equation $z^7 - 2iz^4 - iz^3 - 2 = 0$. 5

(ii) Find the inverse of the matrix if it exists by performing suitable row operations on the augmented matrix $[A : I]$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$ 5

(c) (i) If $f : X \to Y$ be a map and $B \subseteq Y$, then prove $f^{-1}(B^C) = (f^{-1}(B))^C$. 4

(ii) $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, where $n \in \mathbb{N}$. Find $\bigcup_{n\in N} A_n$ and $\bigcap_{n\in N} A_n$. 2

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

Find $f^{-1}(1), f^{-1}(-1), f^{-1}([0, 1])$. 4

(d) (i) State the induction principle and use it to show that for any positive integer $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. 4

(ii) Write as an implication 'square of an even number is divisible by 4'. Then use direct proof to prove it. 3

(iii) Give proof using contrapositive 'For an integer x if x^2-6x+5 is even, then x is odd'. 3

(e) (i) Use the invertible matrix theorem to decide if A is invertible

$$A=\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \\ -5 & -1 & 9 \end{bmatrix}$$ 2

(ii) Compute $det\ A$ where

$$A=\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$ 4

(iii) What do you mean by equivalence class for an equivalence relation? For the relation $a \equiv b \ mod(5)$ on z, find all the distinct equivalence classes of z. 1+3=4

(f) (i) Solve the system of equations

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

4

(ii) Choose h and k such that the system has 4

(a) no solution

(b) a unique solution

(c) many solutions

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

(iii) Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form. 2

(g) (i) Prove that the indexed set $S=\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\vec{v}_1 \neq \vec{0}$, then some $\vec{v}_j$ (with $j>1$) is a linear combination of the preceding vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_{j-1}$. 5

(ii) Use Cramer's rule to compute the solutions of the system 3

$$-5x_1 + 3x_2 = 9$$
$$3x_1 - x_2 = -5$$

(iii) Suppose $T : \mathbb{R}^5 \to \mathbb{R}^2$ and $T(\vec{x}) = A\vec{x}$ for some matrix A and each $\vec{x}$ in $\mathbb{R}^5$.
How many rows and columns does A have? Justify. 2

(h) *(i)* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle ϕ with the counter-clockwise direction taken as positive. Find the standard matrix for this transformation. 3

(ii) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Prove that T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution. 4

(iii) Find the area of the parallelogram whose vertices are $(0, -2)$, $(6, -1)$, $(-3, 1)$ and $(3, 2)$. 3
