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3 (Sem–5 /CBCS) MAT HC 2

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–5026

(Linear Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

   (i) Is $\mathbb{R}^2(\mathbb{R})$ is a subspace of $\mathbb{R}^3(\mathbb{R})$?

   (ii) Let A be a 5×4 matrix. If null space of A is a subspace of $\mathbb{R}^k$ then what is k?

   (iii) Let S be a subset of a vector space $V(F)$ and S contains zero vector of V. Then S is

   (A) linearly independent

   (B) linearly dependent

Contd.

(C) Both linearly independent and linearly dependent

(D) None of the above

(Choose the correct option)

(iv) Write the standard basis of the vector space of polynomial in x with real co-efficient of degree ≤ 3.

(v) "The eigenvalues of a triangular matrix are the entries on its main diagonal." *(State True or False)*

(vi) Define inner product on $\mathbb{R}^n$.

(vii) Which vector is orthogonal to every vector in $\mathbb{R}^n$?

(viii) How do you explain $dim\ W = 1$ geometrically where W is a subspace of the vector space $\mathbb{R}^3(\mathbb{R})$?

(ix) Let A be the 4×4 real matrix,

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & -2 & 0 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

Then the characteristic polynomial for A is

(A) $x^2(x-1)^2$

(B) $(x-1)^2(x+1)^2$

(C) $x^2(x+1)^2$

(D) None of the above

(Choose the correct option)

(x) What do you mean by the length of a vector in $\mathbb{R}^n$?

2. Answer the following questions : $2\times5=10$

(i) Let V be the vector space of all functions from the real field $\mathbb{R}$ to $\mathbb{R}$. Show that $W = \{f : f(7) = 2 + f(1)\}$ is not a subspace of V.

(ii) Show that every subset of an independent set is independent.

(iii) Let $A=\begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}$ and $v=\begin{bmatrix}3\\-2\end{bmatrix}$. Is v a eigenvector of A ?

(iv) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(a,b,c)=(a+b,\ b+c,\ 0)$. Show that the xy-plane $=\{(x,y,0): x,y\in\mathbb{R}\}$ is T-invariant subspace of $\mathbb{R}^3$.

(v) Let $y=\begin{bmatrix}7\\6\end{bmatrix}$ and $u=\begin{bmatrix}4\\2\end{bmatrix}$. Find the orthogonal projection of y onto u.

3. Answer ***any four*** questions : 5×4=20

(i) Prove that the non-zero vectors $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if one of them is a linear combination of the preceding vectors.

(ii) Let $v_1, v_2, \ldots, v_n$ be non-zero eigenvectors of an operator $T: V \to V$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that $v_1, v_2, \ldots, v_n$ are linearly independent.

(iii) Let A and B be two similar matrices of order $n \times n$. Prove that A and B have same characteristic polynomial and hence the same eigenvalues.

(iv) Let $A=\begin{pmatrix}2 & 1 & 0\\0 & 1 & -1\\0 & 2 & 4\end{pmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

(v) Let $A=\begin{bmatrix}7 & 2\\-4 & 1\end{bmatrix}$. Find a formula for A^2, given that $A=PDP^{-1}$ where $P=\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$ and $D=\begin{bmatrix}5 & 0\\0 & 3\end{bmatrix}$.

(vi) Define orthogonal set. If $S=\{u_1, u_2, \ldots, u_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then prove that S is linearly independent and hence is a basis for the subspace spanned by S.

4. *(i)* If a vector space V has a basis $B = \{v_1, v_2, \ldots, v_n\}$, then prove that any set in V containing more than n vectors must be linearly dependent. Also show that every basis of V must consist of exactly n vectors. 5+5=10

OR

Let U and V be vector spaces over the same field. Let $\{u_1, u_2, \ldots, u_n\}$ be a basis of U and let $v_1, v_2, \ldots, v_n$ be any arbitrary vectors in V. Prove that there exists a unique linear mapping $f : U \to V$ such that

$f(u_1) = v_1, f(u_2) = v_2, \ldots, f(u_n) = v_n$ 10

(ii) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. 10

OR

State Cayley-Hamilton theorem for matrices. Use it to express $2A^5 - 3A^4 - A^2 - 4I$ as a linear polynomial in A, when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. 10

(iii) Let T be the linear operator on $\mathbb{R}^3$, defined by

$T(x, y, z) = (2y + z, x - 4y, 3x)$

(a) Find the matrix of T in the basis $\{e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (1,0,0)\}$

(b) Verify that $[T]_e [v]_e = [T(v)]_e$ for any vector $v \in \mathbb{R}^3$. 4+6=10

OR

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen-vectors. 10

(iv) Define orthonormal set and orthonormal basis in $\mathbb{R}^n$. Show that $\{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where

$$u_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

1+1+8=10

OR

Define inner product space. Show that the following is an inner product in $\mathbb{R}^2$:

$\langle u, v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

where $u=(x_1, x_2)$, $v=(y_1, y_2)$.

Also show that for all u, v in $\mathbb{R}^2$

$\|u+v\| \leq \|u\| + \|v\|$ 2+5+3=10