3 (Sem–5/CBCS) MAT HE 1/HE 2/HE 3

2021

(Held in 2022)

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–A

Paper : MAT–HE–5016

(Number Theory)

DSE (H)–1

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

PART–A

1. Choose the correct option : 1×10=10

   (i) Two integers a and b are coprime if there exists some integers x, y such that

   (a) $ax + by = 1$

Contd.

(b) $ax - by = 1$

(c) $(ax + by)^n = 1$

(d) None of the above

(ii) Let $d = gcd(a, b), n \in \mathbb{N}$. If $d \mid c$ and (x_0, y_0) is a solution of linear Diophantine equation $ax + by = c$, then all integral solutions are given by

(a) $(x, y) = \left(x_0 + \frac{bn}{d}, y_0 - \frac{an}{d}\right)$

(b) $(x, y) = \left(x_0 - \frac{bn}{d}, y_0 + \frac{an}{d}\right)$

(c) $(x, y) = \left(x_0 + \frac{an}{d}, y_0 - \frac{bn}{d}\right)$

(d) $(x, y) = \left(x_0 - \frac{an}{d}, y_0 + \frac{bn}{d}\right)$

(iii) A reduced residue system modulo m is a set of integers r_i such that

(a) $[r_i, m] = 1$

(b) $(r_i, m) = 1$

(c) $(r_i, m) \neq 1$

(d) None of the above

(iv) Suppose that m_j are pairwise relatively prime and a_j are arbitrary integers $(j = 1, 2, \ldots k)$ then there exist solution x to the simultaneous congruence $x \equiv a_j \left(mod\ m_j\right)$, such that x are

(a) congruent modulo $M = m_1.m_2.m_3 \ldots m_k$

(b) congruent modulo $M = \sum_{j=1}^{k} m_j$

(c) congruent modulo m_i

(d) Both (a) and (b)

(v) The product of four consecutive positive integers is divisible by

(a) 20

(b) 22

(c) 24

(d) 26

(vi) Euler's ϕ-function of a prime number p, i.e., $\phi(p)$ is

(a) p

(b) $p - 1$

(c) $\frac{p}{2} - 1$

(d) None of the above

(vii) For which value of m,
CRS (mod m) = RRS (mod m) ?

(a) If m is a prime

(b) If m is a composite

(c) If $m < 10$

(d) None of the above

(viii) If $ca \equiv cb(\text{mod}\, m)$, then

(a) $a \equiv b\left(\text{mod}\,\frac{m}{(c, m)}\right)$

(b) $a \equiv b(\text{mod}\, m)$

(c) $a \equiv b(\text{mod}\, m.(c, m))$

(d) None of the above

(ix) The unit place digit of 2^{73} is

(a) 4

(b) 6

(c) 8

(d) 2

(x) The highest power of 7 that divides 50 ! is

(a) 7

(b) 8

(c) 10

(d) 5

2. Answer the following questions : $2\times5=10$

(a) If p is a prime, then prove that $\phi(p!) = (p-1)\,\phi((p-1)!)$ 2

(b) Find all prime number p such that p^2+2 is also a prime. 2

(c) For $n = p^k$, p is a prime, prove that

$$n = \sum_{d|n} \phi(d)$$

where $\sum_{d|n}$ denotes the sum over all positive divisors of n. 2

(d) Find the number of zeros at the end of the product of first 100 natural numbers. 2

(e) Find $\sigma(12)$. 2

3. Answer ***any four*** questions : $5\times4=20$

(a) If ϕ is Euler's phi function, then find $\phi(\phi(1001))$. 5

(b) Find the remainder, when 30^{40} is divided by 17. 5

(c) State and prove Chinese Remainder Theorem. 5

(d) If p_n is the nth prime number, then prove that

$$p_n < 2^{2^{n-1}}$$ 5

(e) If $n = p_1^{k_1} \cdot p_2^{k_2} \, p_3^{k_3} \ldots p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that

(i) $\tau(n) = (k_1+1)(k_2+1)(k_3+1)\ldots(k_n+1)$

(ii) $\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1} \times \frac{p_2^{k_2+1}-1}{p_2-1} \times \ldots \times \frac{p_r^{k_r+1}-1}{p_r-1}$

2½+2½=5

(f) Define Mobius function. Also show that

$$\mu(m \cdot n) = \mu(m) \cdot \mu(n)$$

Hence find $\mu(6)$. 1+3+1=5

PART–B

*Answer **any four** questions :* 10×4=40

4. (a) If $d = (a, n)$, prove that the linear congruence $ax \equiv b \pmod{n}$ has a solution if and only if $d \mid b$. 5

(b) (i) When a number n is divided by 3 it leaves remainder 2. Find the remainder when $3n+6$ is divided by 3. 2

(ii) Prove that $5n+3$ and $7n+4$ are coprime to each other for any natural number n. 3

5. (a) If p is a prime, then prove that

$$(p-1)! \equiv -1 \pmod{p}$$ 5

(b) Using property of congruence show that 41 divides $2^{20}-1$. 5

6. (a) Prove that every positive integer $(n > 1)$ can be expressed uniquely as a product of primes. 5

(b) Determine all solutions in the integers of the Diophantine equation

$$172x + 20y = 1000$$ 5

7. (a) If n be any positive integer and can be expressed as $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, then prove that $\phi(n) = n\prod_{j=1}^{k}\left(1 - \frac{1}{p_j}\right)$. 5

(b) If m and n are any two integers such that $(m, n) = 1$, prove that

$$\phi(m.n) = \phi(m).\,\phi(n).$$ 5

8. (a) For each positive integer $n \geq 1$, show that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$ 5

(b) If k denotes the number of distinct prime factors of positive integer n, then prove that

$$\sum_{d|n} |\mu(d)| = 2^k$$ 5

9. (a) Show that $\sum_{d|n} \mu(d)\,\tau(d) = (-1)^k$

where k denotes the number of distinct prime factors of positive integers n. 5

(b) Prove that

(i) $\tau(n)$ is an odd integer iff n is a perfect square. 3

(ii) For any integer $n \geq 3$, show that

$$\sum_{k=1}^{n} \mu(k!) = 1.$$ 2

10. (a) Let p be an odd prime. Show that the congruence $x^2 \equiv -1 (mod\ p)$ has a solution if and only if $p \equiv 1 (mod\ 4)$. 5

(b) If $n \geq 1$ and $gcd(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 (mod\ n)$. 5

11. (a) If n is a positive integer and p is a prime, then prove that the exponent of the highest power of p that divides $n!$ is $\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$. 5

(b) Solve $3[x] = x + 2\{x\}$ where $[x]$ denotes greatest integer $\leq x$ and $\{x\}$ denotes the fractional part of x. 5

OPTION–B

Paper : MAT–HE–5026

(Mechanics)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (i) What is the physical significance of the moment of a force?

   (ii) State Newton's second law of motion.

   (iii) Define angle of friction.

   (iv) Define the centre of gravity of a body.

   (v) What do you mean by terminal velocity?

   (vi) What is the geometrical representation of the simple harmonic motion?

   (vii) What is the length of arm of a couple equivalent to the couple (P, p) having constituent force of magnitude F?

   (viii) Can a force and a couple in the same plane be equivalent to a single force?

   (ix) Define a couple.

   (x) State Hooke's law.

2. Answer the following questions : 2×5=10

   (a) Find the greatest and least resultant of two forces acting at a point whose magnitudes are P and Q respectively.

   (b) Find the centre of gravity of an arc of a plane curve $y = f(x)$.

   (c) State the laws of static friction.

   (d) Show that impulse of a force is equal to the momentum generated by the force in the given time.

   (e) Write the expression for the component of velocity and acceleration along radial and cross-radial direction for a motion of a particle in a plane curve.

3. Answer ***any four*** questions of the following: 5×4=20

(a) The line of action of a force F divides the angle between its component forces P and Q in the ratio $1:2$. Prove that $Q(F+Q)=P^2$.

(b) P and Q are two like parallel forces. If P is moved parallel to itself through a distance x, show that the resultant of P and Q moves through a distance $\frac{Px}{P+Q}$.

(c) R is the resultant of two forces P and Q acting at a point and at a given angle. If the force P be doubled, show that the new resultant will be of magnitude $\sqrt{2\left(P^2+R^2\right)-Q^2}$.

(d) A particle of mass m moves in a straight line under acceleration mn^2x towards a point O on the line, where x is the distance from O. Show that if $x=a$ and $\frac{dx}{dt}=u$ when $t=0$, then at time t,

$$x=a\cos nt+\frac{u}{n}\sin nt.$$

(e) A particle moving with simple harmonic motion in a straight line has velocity v_1 and v_2 at distance x_1, x_2 from the centre of its path. Show that if T be the period of its motion then $T=2\pi\sqrt{\frac{x_1^2-x_2^2}{v_2^2-v_1^2}}$.

(f) Show that the sum of the kinetic energy and potential energy is constant throughout the motion when a particle of mass m falls from rest at a height h above ground.

4. Answer ***any four*** questions of the following: 10×4=40

(a) Forces P, Q and R act along the sides BC, CA and AB of a triangle ABC and forces P', Q' and R' act along OA, OB and OC, where O is the centre of the circumscribed circle, prove that

(i) $P\cos A+Q\cos B+R\cos C=0$

(ii) $\frac{PP'}{a}+\frac{QQ'}{b}+\frac{RR'}{c}=0$

(b) State and prove Lami's theorem. Forces P, Q and R acting along OA, OB and OC, where O is the circumcentre of triangle ABC, are in equilibrium. Show that

$$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}$$

(c) (i) Find the centre of gravity of a circular arc of radius a which subtends an angle 2α at the centre.

(ii) Find the centre of gravity of a uniform parabolic area cut off by a double ordinate at a distance h from the vertex.

(d) (i) Show that the least force which will move a weight W along a rough horizontal plane is $W \sin\phi$, where ϕ is the angle of friction.

(ii) If a body is placed upon a rough inclined plane, and is on the point of sliding down the plane under the action of its weight and the reactions of the plane only, show that the angle of inclination of the plane to the horizon is equal to the angle of friction.

(e) A particle moves in a straight line under an attraction towards a fixed point on the line varying inversely as the square of the distance from the fixed point. Investigate the motion.

(f) A particle moves in a straight line OA starting from the rest at A and moving with an acceleration which is directed towards O and varies as the distance from O. Discuss the motion of the particle. Hence define simple harmonic motion and time period of the motion.

(g) Find the component of acceleration of a point moving in a plane curve along the initial line and the radius vector. Also find the component of acceleration perpendicular to initial line and perpendicular to radius vector.

(h) A particle is falling under gravity in a medium whose resistance varies as the velocity. Find the distance and velocity at any time t. Also find the terminal velocity of the particle.

OPTION–C

Paper : MAT–HE–5036

(Probability and Statistics)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

(a) Write the sample space for the experiment of tossing a coin three times in succession or tossing three coins at a time.

(b) Is the probability mass function

x	-1	0	1
$P(x)$	0.3	0.4	0.4

admissible? Give reason.

(c) Sketch the area under any probability curve with probability function $p(x)$ between $x=c$ and $x=d$ represented by

$$P(c \le X \le d) = \int_c^d p(x)\,dx.$$

(d) What conclusion one can make about the conditional probability $P(A/B)$ if $P(B)=0$?

(e) State the multiplicative theorem of expectation.

(f) Mention the relationship among the mean, median and mode of the normal distribution.

(g) If X and Y are two independent random variables, then find $Var(2X+3Y)$.

(h) Write the mean and variance of standard normal variate $Z=\frac{X-\mu}{\sigma}$, where μ and σ are mean and standard deviation respectively.

(i) When is the correlation coefficient between two random variables X and Y zero.

(j) State weak law of large number.

2. Answer the following questions : 2×5=10

(a) Prove that probability of any impossible event is zero.

(b) If X is a random variable, then prove that $Var(X) = E(X^2) - \{E(X)\}^2$

(c) Find the constant c such that the function

$$f(x) = cx^2, \; 0 < x < 3$$
$$\qquad 0, \text{ otherwise}$$

is a density function and also find $P(0 < x < 3)$.

(d) A random variable X has density function given by

$$f(x) = 2e^{-2x}, \; x \geq 0$$
$$\qquad 0, \quad \text{otherwise}$$

then find the moment generating function.

(e) Comment on the following statement : "The mean of a binomial distribution is 3 and its standard deviation is 2".

3. Answer ***any four*** parts from the following : 5×4=20

(a) A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give 4 white and the second to give 4 black balls if the balls are not replaced before the second draw.

(b) For two independent events A and B prove that *(i)* A and B are independent, and *(ii)* $\overline{A}$ and $\overline{B}$ are independent.

(c) A random variable X has the function

$f(x) = \dfrac{c}{x^2+1}$, where $-\infty < x < \infty$, then

(i) find the value of the constant c;

(ii) find the probability that X^2 lies between $\frac{1}{3}$ and 1.

(d) The joint probability of two variables X and Y is given by

$$f(x, y) = \frac{1}{42}(2x + y), \; 0 \leq x \leq 2, \; 0 \leq y \leq 3$$
$$\qquad 0 \qquad\qquad , \text{ otherwise}$$

Find *(i)* $F(Y/2)$, and *(ii)* $P(y = 1/x = 3)$

(e) A coin is tossed until a head appears. What is the expectation of the number of tosses required?

(f) The probability of a man hitting a target is $\frac{1}{4}$.

(i) If he fires 7 times, what is the probability of his hitting the target at least twice?

(ii) How many times must he fire so that the probability of his hitting the target at least once is greater than $\frac{2}{3}$?

4. Answer ***any four*** parts from the following: 10×4=40

(a) For n events $A_1, A_2, A_3, \ldots\ldots\ldots, A_n$, prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i-1}^{n} P(A_i) - \sum_{1\le i<j\le n}\sum P(A_i \cap A_j) + \sum_{1\le i<j<k\le n}\sum\sum P(A_i \cap A_j \cap A_k) + \ldots\ldots\ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_n)$$

Hence find $P\left(\bigcup_{i=1}^{3} A_i\right)$

(b) Suppose that two dimensional continuous random variables (X, Y) has joint p.d.f given by

$$f(x, y) = 6x^2y, \; 0 < x < 1, \; 0 < y < 1$$
$$\quad 0 \quad , \text{ elsewhere}$$

(i) Verify that $\int_0^1 \int_0^1 f(x, y)\, dx\, dy = 1$

(ii) Find $P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right)$

(iii) Find $P(X + Y < 1)$

(iv) Find $P(X > Y)$

(v) Find $P(X < 1/Y < 2)$.

(c) *(i)* The probability function of a random variable X is given by

$$f(x, y) = \frac{x^2}{81}, -3 < x < 6$$
$$\quad 0 \;, \text{ otherwise}$$

Find the probability density function for the random variable function $u = \frac{1}{3}(12 - X)$.

Contd.

(ii) Define moment generating function of a random variable X. Find the moment generating function of binomial distribution.

(d) (i) The probability curve $y = f(x)$ has a range from 0 to ∞. If $f(x) = e^{-x}$, find the mean and variance.

(ii) If X be a continuous random variable with probability density function

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

compute $P(X \le 1.5)$

(e) (i) Derive Poisson distribution as a limiting case of binomial distribution.

(ii) Prove that mean and variance of a binomially distributed variable are respectively np and npq.

(f) (i) Define correlation coefficient of two random variables X and Y. Show that correlation coefficient is independent of change of origin and scale.

(ii) Obtain the equation of two lines of regression for the following data :

X :	65	66	67	67	68	69	70	72
Y :	67	68	65	68	72	72	69	71

Also obtain the estimate of X for $Y = 70$.

(g) (i) If X is a random variable with mean μ and variance σ^2, then for any positive number k, prove that

$$P\{|X - \mu| \ge k\sigma\} \le \frac{1}{k^2}$$

(ii) A symmetric die is thrown 600 times. Find the lower bound for the probability of getting 80 to 120 sixes.

(h) The random variables X and Y have the following joint probability density function :

$$f(x, y) = \begin{cases} 2 - x - y; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find—

(i) marginal probability density functions of X and Y;

Contd.

(ii) conditional density functions ;

(iii) $Var(X)$;

(iv) $Var(Y)$;

(v) covariance between X and Y.
