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**3 (Sem-6/CBCS) MAT HC 1 (N/O)**

**2024**

**MATHEMATICS**

(Honours Core)

Paper : MAT-HC-6016

New Syllabus

***(Riemann Integration and Metric Spaces)***

Full Marks : 80

Time : Three hours

Old Syllabus

***(Complex Analysis)***

Full Marks : 60

Time : Three hours

***The figures in the margin indicate  
full marks for the questions.***

Contd.



New Syllabus

**(Riemann Integration and Metric Spaces)**

Full Marks : 80

Time : Three hours

1. Answer the following as directed :

1×10=10

(a) Let  $f: [a, b] \rightarrow R$  be a bounded function and  $P, Q$  are partitions of  $[a, b]$ . If  $Q$  is a refinement of  $P$ , then

(i)  $L(f, Q) \leq L(f, P)$

(ii)  $U(f, P) \leq U(f, Q)$

(iii)  $U(f) \leq L(f)$

(iv)  $L(f) \leq U(f)$

(Choose the correct option)

(b) Find the value of  $\int_0^{\infty} e^{-x} dx$ .

(c) Show that  $\Gamma(1) = 1$ .

(d) Define Cauchy sequence in a metric space.

(e) State whether the following statement is true **or** false :

“Each subset of a discrete metric space is open.”

(f) If the mapping  $d: R^2 \times R^2 \rightarrow R$  is defined as  $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|$ , then which one of the following statements is true ?

(i)  $d$  is the usual metric on  $R^2$

(ii)  $d$  is uniform metric on  $R^2$

(iii)  $d$  is a pseudo metric on  $R^2$

(iv) None of the above statements is true

(g) Which of the following statements is not true ?

(i) In a metric space countable union of open sets is open

(ii) In a metric space finite union of closed sets is closed

(iii) A non-empty subset of a metric space is closed if and only if its complement is open

(iv) None of the above statements is true

(h) When is a metric space said to be connected.



- (i) State whether the following statement is true **or** false :

“Image of an open set under a continuous function is open.”

- (j) Under what condition the metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  are said to be equivalent?

2. Answer the following questions :  $2 \times 5 = 10$

- (a) Let  $u, v : [a, b] \rightarrow R$  be differentiable and  $u', v'$  are integrable on  $[a, b]$ . Then show that

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx.$$

- (b) Show that a subset  $F$  of a metric space  $(X, d)$  is closed if and only if  $\bar{F} = F$ .

- (c) Let  $(Y, d_Y)$  be a subspace of a metric space  $(X, d_X)$  and  $S_X(z, r)$  and  $S_Y(z, r)$  are open balls with center at  $z \in Y$  and radius  $r$  in the metric space  $(X, d_X)$  and  $(Y, d_Y)$  respectively.

Prove that  $S_Y(z, r) = S_X(z, r) \cap Y$ .

- (d) Show that the image of a Cauchy sequence under uniformly continuous function is again a Cauchy sequence.

- (e) Show that a contraction mapping on a metric space is uniformly continuous.

3. Answer **any four** questions :  $5 \times 4 = 20$

- (a) Show that a bounded function  $f : [a, b] \rightarrow R$  is integrable if and only if for each  $\epsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \epsilon$ .

- (b) Let  $g$  be a continuous function on the closed interval  $[a, b]$  and the function  $f$  be continuously differentiable on  $[a, b]$ . Further if  $f'$  does not change sign on  $[a, b]$ , then show that there exists  $c \in [a, b]$  such that

$$\int_a^b f(x)g(x)dx = f(a) \int_a^c g(x)dx + f(b) \int_c^b g(x)dx.$$



(c) Let  $(X, d)$  be a metric space and the function  $d^*: X \times X \rightarrow \mathbb{R}$  is defined as

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X$$

Show that  $(X, d^*)$  is a bounded metric space.

(d) Let  $Y$  be a non-empty subset of the metric space  $(X, d)$ . Prove that the subspace  $(Y, d_Y)$  is complete if and only if  $Y$  is closed on  $(X, d)$ .

(e) Show that composition of two uniformly continuous functions is also uniformly continuous.

(f) Show that a metric space  $(X, d)$  is disconnected if and only if there exists a continuous function of  $(X, d)$  onto the discrete two element space  $(X_0, d_0)$ , i.e.,  $X_0 = \{0, 1\}$  and  $d_0$  is the discrete metric on  $X_0$ .

4. Answer the following questions :  $10 \times 4 = 40$

(a) Let  $f$  be a function on an interval  $J$  with  $n$ th derivative  $f^{(n)}$  continuous on  $J$ . If  $a, b \in J$ , then show that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \dots + \frac{f^{(n)}(a)}{(n-1)!}(b-a)^{n-1} + R_n$$

$$\text{where, } R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t) dt$$

**Or**

Let  $f: [0, 1] \rightarrow \mathbb{R}$  be continuous and

$$c_i \in \left[ \frac{i-1}{n}, \frac{i}{n} \right], n \in \mathbb{N}.$$

Then show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(c_i) = \int_0^1 f(x) dx.$$

Hence show that

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r}{r^2 + n^2} = \log \sqrt{2}.$$

5+5=10



- (b) Let  $l_p$  ( $p \geq 1$ ) be the set of all sequences of real numbers such that if

$$x = \{x_n\}_{n \geq 1} \in l_p, \text{ then } \sum_{i=1}^{\infty} |x_i|^p < \infty.$$

Prove that the function  $d : l_p \times l_p \rightarrow \mathbb{R}$

$$\text{defined by } d(x, y) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^p \right\}^{\frac{1}{p}}$$

is a metric on  $l_p$ . Also show that  $l_p$  is a complete metric space. 4+6=10

**Or**

- (i) Let  $(X, d)$  be a metric space and  $\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1}$  be two sequences in  $X$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  as  $n \rightarrow \infty$ . Then show that  $d(x_n, y_n) \rightarrow d(x, y)$  as  $n \rightarrow \infty$ .

4

- (ii) Let  $(X, d)$  be a metric space and  $Y$  a subspace of  $X$ . Let  $Z$  be a subset of  $Y$ . Then show that  $Z$  is closed in  $Y$  if and only if there exists a closed set  $F \subseteq X$  such that  $Z = F \cap Y$ .

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- (c) What are meant by contraction mapping and fixed point of a contraction mapping in a metric space? If  $T : X \rightarrow X$  is a contraction mapping on a complete metric space, then show that  $T$  has a unique fixed point. (1+1)+8=10

**Or**

If  $(X, d)$  be a metric space, then show that the following statements are equivalent :

- (i)  $(X, d)$  is disconnected.
- (ii) There exist two non-empty disjoint subsets  $A$  and  $B$ , both open in  $X$ , such that  $X = A \cup B$ .
- (iii) There exist two non-empty disjoint subsets  $A$  and  $B$ , both closed in  $X$ , such that  $X = A \cup B$ .
- (iv) There exists a proper subset of  $X$  that is both open and closed in  $X$ .



(d) (i) Let  $f : [a, b] \rightarrow R$  be integrable and

$$F(x) = \int_a^x f(t) dt; \quad x \in [a, b]. \quad \text{Show}$$

that  $F$  is continuous on  $[a, b]$ . Also show that  $F$  is differentiable at  $x \in [a, b]$  if  $f$  is continuous at  $x \in [a, b]$  and  $F'(x) = f(x)$ . 7

(ii) Let  $(X, d)$  be a metric space and  $\rho : X \times X \rightarrow R$  be defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}; \quad x, y \in X.$$

Show that  $d$  and  $\rho$  are equivalent metrics. 3

**Or**

(iii) Show that a subset  $G$  of a metric space  $(X, d)$  is open if and only if it is the union of all open balls contained in  $G$ . 5

(iv) Give example, with justification, of a homeomorphism from a metric space onto another metric space which is not an isometry. 5

Old Syllabus

Full Marks : 60

**(Complex Analysis)**

Time : Three hours

1. Answer the following as directed :  $1 \times 7 = 7$

(a) Determine the accumulation point of the set  $z_n = \frac{i}{n}$  ( $n = 1, 2, 3, \dots$ )

(b) Describe the domain of  $f(z) = \frac{z}{z + \bar{z}}$ .

(c) Define an entire function.

(d) Determine the singular points of

$$f(z) = \frac{2z + 1}{z(z^2 + 1)}$$

(e) The value of  $\log e$  is

(i) 1

(ii)  $1 + 2n\pi i$

(iii)  $2n\pi i$

(iv) 0

(Choose the correct option)



(f)  $\lim_{n \rightarrow \infty} \left( -2 + i \frac{(-1)^n}{n^2} \right)$  is equal to

- (i) 0                      (ii) -2  
(iii)  $-2 + i$               (iv) limit does not exist  
(Choose the correct option)

(g) The power expression for  $\cos z$  is

- (i)  $\frac{e^z + e^{-z}}{2}$                       (ii)  $\frac{e^{iz} + e^{-iz}}{2}$   
(iii)  $\frac{e^{iz} + e^{-iz}}{2i}$                       (iv)  $\frac{e^z - e^{-z}}{2}$   
(Choose the correct option)

2. Answer the following questions:  $2 \times 4 = 8$

(a) Sketch the set

$$|z - 1 + i| \leq 1$$

(b) Prove that  $f'(z)$  exists every where for the function  $f(z) = iz + 2$ .

(c) If  $f(z) = \frac{z}{\bar{z}}$ , prove that  $\lim_{z \rightarrow 0} f(z)$  does not exist.

(d) Evaluate  $\int_1^2 \left( \frac{1}{t} - i \right)^2 dt$ .

3. Answer **any three** questions from the following:  $5 \times 3 = 15$

(a) (i) Show that if  $e^z$  is real, then  
 $\text{Im} z = n\pi$  ( $n = 0, \pm 1, \pm 2, \dots$ )      3

(ii) Show that  $\exp(2 \pm 3\pi i) = -e^2$ .      2

(b) Suppose that  $f(z) = u(x, y) + iv(x, y)$ , where  $z = x + iy$  and  $z_0 = x_0 + iy_0$ ,  $w_0 = u_0 + iv_0$ . Then prove that

$$\lim_{z \rightarrow z_0} f(z) = w_0 \text{ if}$$

$$\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0 \text{ and}$$

$$\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0$$

(c) Show that  $f'(z)$  exists everywhere, when  $f(z) = e^z$ .

(d) Evaluate  $\int_C \frac{dz}{z}$ , where  $C$  is the top half of the circle  $|z| = 1$  from  $z = 1$  to  $z = -1$ .



(e) Let  $C$  denote the positively oriented boundary of the square whose sides lie along the lines  $x = \pm 2$  and  $y = \pm 2$ . Applying Cauchy's integral formula,

$$\text{evaluate } \int_C \frac{e^{-z} dz}{z - (\pi i/2)}.$$

4. Answer **either** (a) **or** (b) and (c):

(a) Suppose that  $f(z) = u(x, y) + iv(x, y)$  and that  $f'(z)$  exists at a point  $z_0 = x_0 + iy_0$ . Prove that the first order partial derivatives of  $u$  and  $v$  must exist at  $(x_0, y_0)$  and they must satisfy the Cauchy-Riemann equations  $u_x = v_y$  and  $u_y = -v_x$  there.

Also show that  $f'(z) = u_x + iv_x = v_y - iv_y$  where partial derivatives are to be evaluated at  $(x_0, y_0)$ . 10

**Or**

(b) If  $z_0$  and  $w_0$  are points in the  $z$ -plane and  $w$ -plane respectively, then prove that  $\lim_{z \rightarrow z_0} f(z) = \infty$  if and only if

$$\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$$

Hence show that  $\lim_{z \rightarrow -1} \frac{iz + 3}{z + 1} = \infty$

$$4+2=6$$

(c) If  $w = f(z) = \bar{z}$ , examine whether

$$\frac{dw}{dz} \text{ exists or not.} \quad 4$$

5. Answer **either** (a) **or** (b):

(a) Let  $C$  denote a contour of length  $L$  and suppose that a function  $f(z)$  is piecewise continuous on  $C$ . If  $M$  is a non-negative constant such that  $|f(z)| \leq M$  for all points  $z$  on  $C$  at which  $f(z)$  is defined, then prove that

$$\left| \int_C f(z) dz \right| \leq ML$$

Hence show that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \pi/3, \text{ where } C \text{ is the arc of}$$

the semicircle  $|z| = 2$  from  $z = 2$  to  $z = 2i$  that lies in the first quadrant. 10

**Or**

(b) State and prove Liouville's theorem.



6. Answer **either** (a), (b), (c) **or** (d):

(a) Prove that if a series of complex numbers converges, then the  $n$ th term converges to zero as  $n$  tends to infinity. 3

(b) Test the convergency of the sequence

$$z_n = \frac{1}{n^3} + i \quad (n = 1, 2, \dots) \quad 3$$

(c) Find Maclaurin's series for the entire function  $f(z) = l^z$ . 4

**Or**

(d) Suppose that a function  $f$  is analytic throughout a disc  $|z - z_0| < R_0$  centred at  $z_0$  and with radius  $R_0$ . Then prove that  $f(z)$  has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0)$$

$$\text{where } a_n = \frac{f^n(z_0)}{n!} \quad (n = 0, 1, 2, \dots)$$