Total number of printed pages–20

3 (Sem–6/CBCS) MAT HE 5/6/7

2024

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

OPTION–F

(Group Theory–II)

Paper : MAT–HE–6066

OPTION–G

(Mathematical Finance)

Paper : MAT–HE–6076

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

1. Answer the following questions : 1×10=10

(a) Write down the moment of inertia of a circular disc of mass M and radius a about a diameter.

(b) Define the principal axes of a rigid body at a point O of the body.

(c) Define the point of suspension.

(d) Define equimomental systems.

(e) What is the formula for calculating moment of inertia for a point mass?

(f) Can the parallel axes theorem be applied to any shape or object?

(g) What is meant by a conservative mechanical system?

(h) A particle moves on the surface of a sphere. What is the degree of freedom of the particle?

(i) Define the centre of percussion.

(j) Define radius of gyration of a rigid body about a line.

2. Answer the following questions : 2×5=10

(a) A rigid body of mass 5 units rotates with angular velocity $\bar{\omega} = (1, -1, 1)$ and has angular momentum $\bar{\Omega} = (2, -3, 1)$. Calculate the K.E. of the body.

(b) A rigid body with one point fixed rotates with angular velocity $\bar{\omega}$ and has angular momentum $\bar{\Omega}$. Derive the equation for kinetic energy in terms of $\bar{\omega}$ and $\bar{\Omega}$.

(c) Consider a rigid body consisting of four particles m_1, m_2, m_3, m_4 having masses 2 units, 3 units, 4 units, 5 units and located at the points (–1, 0, 1), (0, 0, 1), (1, 1, 0), (1, 0, 1) respectively. Find the products of inertia about

(i) x-axis and z-axis,

(ii) y-axis and z-axis.

(d) The length AB and AD of the sides of a rectangle $ABCD$ of mass M are $2a$ and $2b$. Obtain the product of inertia of the rectangle about AB–AD.

(e) A particle of mass 3 units is located at the point (2, 0, 0). The particle rotates about O with angular velocity $\vec{\omega} = \hat{k}$. Find the angular momentum of the particle about O.

3. Answer ***any four*** questions : 5×4=20

(a) Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being a and b.

(b) Show that the momental ellipsoid at the centre of an elliptic plate is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \text{constant}.$$

(c) Find the principal axes at a corner of a unit square.

(d) A solid homogeneous cone of height h and vertical angle 2α, oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is

$$\frac{h}{5}\left(4 + \tan^2 \alpha\right)$$

(e) Obtain the Lagrangian for a simple pendulum and hence derive the equations of motion of the body.

(f) Show that the momental ellipsoid at any point O of a material straight rod AB of mass M and length $2a$ represents a right circular cylinder having the rod as axis.

4. Answer ***any four*** parts : 10×4=40

(a) (i) State D'Alembert's principle and hence obtain the general equations of motion of a rigid body in vector form. 5

(ii) Two persons are situated on a perfectly smooth horizontal plane at a distance a from each other. One of the persons of mass M throws a ball of mass m towards the other which reaches him in time t. Prove that the first person will begin to slide along the plane with velocity $\frac{ma}{Mt}$. 5

(b) (i) Show that centre of suspension and centre of oscillation are convertible. 5

Contd.

(ii) An elliptical lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation. Prove that the eccentricity is $\frac{1}{2}$. 5

(c) (i) Find the time of oscillation of a compound pendulum, consisting of a rod of mass m and length a carrying at one end a sphere of mass m_1 and diameter $2b$, the other end of the rod being fixed. 5

(ii) State and prove the theorem on conservation of energy. 5

(d) (i) Find the centre of percussion of a triangle ABC which is free to move about its side BC. 5

(ii) Show that for a thin hemispherical shell of radius a and mass M, the principal moments of inertia at the centre of gravity are $\frac{5}{12}Ma^2$, $\frac{5}{12}Ma^2$, $\frac{2}{3}Ma^2$. 5

(e) (i) State and prove the parallel axes theorem. 5

(ii) An elliptic area of eccentricity e is rotating with angular velocity ω about one latus rectum. Suddenly this latus rectum is loosed and the other fixed. Show that the new angular velocity is $\omega\frac{1-4e^2}{1+4e^2}$. 5

(f) A uniform rod OA of length $2a$, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O and is inclined at a constant angle α to OZ. Show that the value of α is either zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

(g) A uniform solid cylinder is placed with its axis horizontal on a plane, whose inclination to the horizon is α. Show that the least co-efficient of friction between it and the plane, so that it may roll and not slide is $\frac{1}{3}\tan\alpha$. If the cylinder be hollow and of small thickness, the least value is $\frac{1}{2}\tan\alpha$. 8+2=10

(h) (i) Define impressed forces and effective forces. 2

(ii) Show that the centre of inertia of a body moves as if all the mass of the body were collected at it, and as if all the external forces acting in the body were acting on it in directions parallel to those in which they act. 8

OPTION–F

(Group Theory–II)

Paper : MAT–HE–6066

1. Answer the following questions as directed : $1\times10=10$

(a) Is the mapping $\phi:\mathbb{R}\to\mathbb{R}$ such that $\phi(x)=x^3$ an isomorphism under addition ?

(b) Define automorphism of a group.

(c) Find the order of (1, 2) in $\mathbb{Z}_2\oplus\mathbb{Z}_4$.

(d) Is the group $\mathbb{Z}_2\oplus\mathbb{Z}_3$ Abelian ?

(e) Mention one normal subgroup of the symmetric group S_5.

(f) Let G be a group and let $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, then G is .... *(Complete the statement)*

(g) If G is a group of order pq, where p and q are not necessarily distinct primes, then order of $Z(G)$ is

(i) 1

(ii) pq

Contd.

(iii) either 1 or pq

(iv) neither 1 nor pq

(Choose the correct option)

(h) Define Sylow p-subgroup of a finite group.

(i) The conjugacy relation on a group is not a transitive relation.

(State True or False)

(j) Find the Sylow 2-subgroups of the symmetric group S_3.

2. Answer the following questions : 2×5=10

(a) Let G be a group and $a \in G$. Prove that the mapping of ϕ_a defined by $\phi_a(x) = axa^{-1}$ is an automorphism of G.

(b) Prove or disprove that $\mathbb{Z} \oplus \mathbb{Z}$ is a cyclic group.

(c) Find the order of the element $5 + \langle 6 \rangle$ in the factor group $\mathbb{Z}_{18} / \langle 6 \rangle$.

(d) Is the group $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$? Justify your answer.

(e) Is a group of order 35 is cyclic ? Give reason.

3. Answer ***any four*** questions : 5×4=20

(a) Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. Prove that G is Abelian if and only if $\overline{G}$ is Abelian.

(b) Let G and H be two groups. Show that $G \oplus H$ is Abelian if and only if both G and H are Abelian.

(c) Let G be a finite Abelian group and let p be a prime that divides the order of G. Show that G has an element of order p.

(d) Let us consider the subgroup $H = \{(1), (12)(34)\}$ of the alternating group A_4. Is H a normal subgroup of A_4 ? Justify your answer.

(e) Show that any Abelian group of order 45 has an element of order 15. Does every Abelian group of order 45 have an element of order 9 ? Justify. 3+2=5

(f) Let G be a finite group and let p be a prime number. Prove that if p^k divides order of G, then G has at least one subgroup of order p^k.

4. Answer the following questions : 10×4=40

(a) Define automorphism of a group. Let $G = SL(2, \mathbb{R})$ be the group of 2×2 real matrices with determinant 1 and let M be any 2×2 real matrix with determinant 1. Show that the mapping $\phi_M : G \to G$ defined by $\phi_M(A) = MAM^{-1}$ for all A in G is an automorphism of G.

Let H be any group. Prove that the mapping $\alpha(g) = g^{-1}$ for all g in G is an automorphism of G if and only if G is Abelian. 2+3+5=10

Or

Let $G_1, G_2, \ldots, G_n$ be finite cyclic groups. Prove that $G_1 \oplus G_2 \oplus \ldots \oplus G_n$ is cyclic if and only if $|G_i|$ and $|G_j|$ are relatively prime when $i \neq j$. Hence show that $\mathbb{Z}_2 \oplus \mathbb{Z}_{15}$ is cyclic. 7+3=10

(b) Prove that every group is isomorphic to a group of permutations. 10

Or

Let G be a group with center $Z(G)$ and let $\mathrm{Inn}(G)$ be the set of all inner automorphisms of G. Prove that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$. Hence show that $\mathrm{Inn}(D_6)$ is isomorphic to D_3. 7+3=10

(c) Define internal direct product of a finite number of subgroups of a group.

Express U(105) as an internal direct product of proper subgroups in three different ways.

Let $\mathbb{R}^*$ be the multiplicative group of all nonzero real numbers and let $\mathbb{R}^+$ be the multiplicative group of all positive real numbers. Show that $\mathbb{R}^*$ is the internal direct product of $\mathbb{R}^+$ and the subgroup $\{1, -1\}$. 2+3+5=10

Or

Prove that if H is a subgroup of a finite group G and order of H is a power of a prime p, then H is contained in some Sylow p-subgroup of G. 10

(d) State and prove Sylow's 3rd theorem. Using it show that every group of order 200 has a proper nontrivial normal subgroup. 6+4=10

Or

Let G be a finite Abelian group of order $p^n m$ where p is a prime that does not divide m. Prove that $G = H \times K$ and

$|H| = p^n$, where $H = \left\{x \in G \mid x^{p^n} = e\right\}$

and $K = \left\{x \in G \mid x^m = e\right\}$.

Show that there are two Abelian groups of order 108 that have exactly one subgroup of order 3. 7+3=10

OPTION–G

(Mathematical Finance)

Paper : MAT–HE–6076

1. Answer the following as directed: 1×10=10

(a) Does bank participate in OTC market?

(b) What do you mean by LIBID?

(c) What is hedging?

(d) What do you mean by fixed-for-floating swap?

(e) Write the full form of LIBOR.

(f) What is short hedge?

(g) The difference between the price at which the securities are sold and the price at which they are repurchased is the interest it earns. Then the interest rate is referred to as ______.

(Fill in the blank)

(h) What do you mean by continuous compounding?

(i) A ______ is the single discount rate that when applied to all cash flows, gives a bond price equal to its market price. *(Fill in the blank)*

(j) What is liquidity?

2. Answer the following : 2×5=10

(a) What is meant by the 'gamma' of a stock option?

(b) A bank quotes an interest rate of 5% per annum with quarterly compounding. What is the equivalent rate with continuous compounding?

(c) Why does a loan in the repo market involve very little credit risk?

(d) Write the names of 'three' broad categories of traders.

(e) What is the difference between the forward price and the value of a forward contract?

3. Answer ***any four*** parts : 5×4=20

(a) An investor receives Rs. 1500 in one year in return for an investment of Rs. 1200 new. Find the percentage of return per annum with

(i) annual compounding;

(ii) semiannual compounding.

(b) Explain the principle of risk-neutral valuation.

(c) Write short notes on

(i) Bond pricing;

(ii) Forward rates.

(d) The risk-free rate of interest is 7% per annum with continuous compounding, and the dividend yield on a stock index is 3·2% per annum. The current value of the index is 150. What is the six month futures price?

(e) What is the difference between buying a put option and writing a cell option? Draw their pay off diagrams as well.

(f) A loan of Rs 20,000 has been issued for 10 years. Compute the amount to be repaid by the borrower to the lender if simple interest is charged @ 8·5% per year.

4. Answer ***any four*** parts : 10×4=40

(a) A man deposits a bank of Rs 25,000 at the end of each year, for 10 years. If the rate of interest is 12% per annum compounding annually, what would be the sum standing to his credit at the end of that period?

(b) Name six factors that affect stock option prices. Explain *any three* of them.

(c) Explain the following terms :

(i) Shorting

(ii) Spot rates

(iii) Forward rates

(iv) Short rate

(d) Discuss the Black-Scholes formula for European options.

(e) Suppose that zero interest rates with continuous compounding are as follows :

Maturity (monthly)	*Rate (% per annum)*
3	5
6	5·4
9	5·8
12	6·2
15	6·4

Calculate forward interest rates for the 2nd, 3rd and 4th quarters.

(f) A stock price is currently Rs. 40. It is known that at the end of 4 months it will be either Rs. 75 or Rs. 48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2 month European call option with a strike price of Rs 49? Use no-arbitrage arguments.

(g) A company had granted 5,00,000 options to its executives. The stock price and strike price are both Rs. 40. The options last for 12 years and vast after 4 years. The company decides to value the options using an expected life of 5 years and a volatility of 30% per annum. The company pay no dividends and the risk-free rate is 4%. What will be the company report as an expense for the options on its income statement ?

(h) A bank can borrow at LIBOR. The two month LIBOR rate is 0·28% per annum with continuous compounding. Assuming that interest rates cannot be negative, what is the arbitrage opportunity if the three month LIBOR rate becomes without an arbitrage opportunity being created ?
