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Fractal Structure in Gauge Fields

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Abstract: In this work, we investigate fractal properties in Yang–Mills fields, in particular their Hausdorff fractal dimension. Fractal properties of quantum chromodynamics (QCD) have been suggested as the origin of power-law distributions in high energy collisions, as well as of non-extensive properties that have been observed experimentally. The fractal dimension obtained here can be calculated directly from the properties of the field theory.

Keywords: Yang-Mills fields; renormalization; scaling properties; non extensive statistics; fractal structure; fractal dimension; effective coupling; QCD

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Yang–Mills field theory unifies three of the four known forces in a single framework in what is known as the Standard Model. For the two weakest forces, the results obtained with the theory describe the experimental data accurately. For QCD, although the same success is expected, calculation difficulties becomes prohibitive. Recently, a new approach was proposed using the general scaling properties of Yang–Mills field theory (YMF) to obtain recurrence formulas that allow calculating QCD processes at high orders of perturbation. This was done by identifying fractal structures of YMF, which may give rise to non-extensivity [1,2]. In the present work, we study some properties of the fractal structures in that important field theory, as its fractal dimension.

Our starting point is the quantity:

$$Z = \mathrm{Tr}\, \langle \Psi_o | \, e^{iHt} \, | \Psi_o \rangle \tag{1}$$

where Ψ_o is an autovector of the Hamiltonian H and that plays the role of the partition function. One can write the Hamiltonian operator as:

$$H = H_o + H_i \,, \tag{2}$$

with H_o being the Hamiltonian operator for a *free* effective parton, and H_i represents the interaction operator among the fields. Introducing the states:

$$|\Psi_n\rangle = \prod_{j=1}^{n} g e^{iH_o(t_j - t_{j-1})} \, |\Psi_o\rangle \ , \tag{3}$$

where t_0 is the initial time, before which interaction among the fields is neglected, g is the effective coupling, and:

$$|\psi_N\rangle = |(\gamma_1, m_1, p_1), \ldots, (\gamma_N, m_N, p_N)\rangle \ , \tag{4}$$

we can write:

$$|\Psi(t)\rangle = \sum_n \sum_N \langle \Psi_n|\Psi(t)\rangle \langle \Psi_N|\Psi_n\rangle \langle (\gamma_1, m_1, p_1), \ldots, (\gamma_N, m_N, p_N)|\psi_N\rangle \times \\ |(\gamma_1, m_1, p_1), \ldots, (\gamma_N, m_N, p_N)\rangle \ . \tag{5}$$

Here, (γ_i, m_i, p_i) represents, respectively, the set of quantum numbers, the effective mass, and the four-momentum of the effective parton.

The states $|\Psi_n\rangle$ are eigenstates of the Hamiltonian operator, H, with a well-defined number of interactions, n. Of course, the time-dependent state $|\Psi\rangle$ can be spanned in terms of $|\Psi_n\rangle$, and the matrix element $\langle\Psi_n|\Psi\rangle$ determines the probability that at instant t, the number of interactions since the initial time, t_0, is n, which follows from a perturbative expansion of the interaction operator. The probability of n interactions depends on the intensity of interaction, G, and the stronger is the intensity of interaction, the larger is the average number of interactions, so we write:

$$\langle\Psi_n|\Psi\rangle = G^n P(E)dE \,, \tag{6}$$

where we consider that the total energy of the N-particle system is between E and $E + dE$, with probability $P(E)dE$.

The states $|\psi_N\rangle$ are states of N effective partons, which do not interact, so they are eigenstates of H_o. The number of partons, N, and the number of interactions, n, are not necessarily equal, since some states with N particles are produced in high order interactions. The matrix element $\langle\psi_N|\Psi_n\rangle$ gives the probability that, after n interactions, the system will have N effective partons. This probabilities depends only on n and N, and we write:

$$\langle\psi_N|\Psi_n\rangle = C_N(n) \,, \tag{7}$$

where $\sum_n C_N(n) = 1$.

Since $|\psi_N\rangle$ are eigenstates of H_o, they can be considered as a system of N non-interacting particles, so:

$$|\psi_N\rangle = \sum_{\{n\}} \mathcal{S} \, |(\gamma_1, m_1, p_1), \ldots, (\gamma_N, m_N, p_N)\rangle \ , \tag{8}$$

where $\mathcal{S}$ is the (anti)symmetrization operator and (γ_i, m_i, p_i) are, respectively, the set of relevant quantum numbers to identify the parton, its effective mass, and its momentum. The sum is performed over all possible configurations of n interactions, with n from one to infinity. Notice, however, that for small n, most of the elements $\langle\Psi_n|\psi_N\rangle$ are null, since there is a minimum number of interactions necessary to generate N particles.

In particular, we are interested in states where one of the particles has a particular state, (γ_o, m_o, p_o), all the other $N-1$ particles occupying any possible state. The number of possible ways that the system can evolve to reach this particular configuration is infinity, since it can be achieved by any sufficiently large number of interactions, N. Systems where a particular configuration can be reached by a large number of possible processes are usually considered statistically, and we do the same here. This, in fact, is the procedure used, for instance, in lattice QCD [3]. The probability for the configuration is given by $P(p_o^0) = \langle(\gamma_o, m_o, p_0), \ldots|\psi_N\rangle$ with the bracket being calculated statistically, as for an ideal gas with N particles. Following the usual methods for counting states of statistical mechanics [4] and using the fact that the effects of the operator $\mathcal{S}$ are small, this results in:

$$\langle(\gamma_o, m_o, p_0), \ldots|\psi_N\rangle = \frac{1}{8\pi}\frac{\Gamma(4N)}{\Gamma(4N-4)}\left(1 - \frac{p_o^0}{E}\right)^{4N-5} d^4(p_o/E) \,. \tag{9}$$

where E is the total energy of the system. We introduce the quantity λ, which is the energy per degree of freedom and $\lambda \geq p_o^0$, with p_o^0 being the energy of the parton with index o. For large values of N, we can write:

$$\left(1 - \frac{p_o^0}{E}\right)^{4N-5} = \left(1 + \frac{p_o^0}{E}\right)^{-(4N-5)} \exp\left[\frac{-1}{4N-5}\left(\frac{p_o^o}{\lambda}\right)^2\right], \tag{10}$$

and the exponential factor is approximately equal to the unit for N sufficiently large, so:

$$\left(1 - \frac{p_o^0}{E}\right)^{4N-5} = \left(1 + \frac{p_o^0}{E}\right)^{-(4N-5)}. \tag{11}$$

Considering the discussion above, we have that the probability to find the system in a configuration with N particles and one particle in the state $\gamma_o, m_o, p_o)$ is $P(\varepsilon_o)d\varepsilon_o = \langle(\gamma_o, m_o, p_0), \ldots |\psi_N\rangle$, with the probability density given by:

$$P(\varepsilon_o) = \sum_n \sum_N G^n \left(\frac{N}{n(\tilde{N}-1)}\right)^4 \left(1 - \frac{\varepsilon_o}{E}\right)^{4N-5} P(E). \tag{12}$$

The probability density $P(E)$ refers to the possibility of the fluctuation of the system energy, which results from the fact that the main system is itself a parton, as will be clear below.

So far, we have considered the system as an ideal gas, but this is not completely correct for effective partons, since they have an internal structure, which is represented by self-interaction. In Yang–Mills field theory, self-interaction is taken into account by renormalization of the fields after divergences are cut off. Such renormalization is possible only because of scaling properties of the theory, which are represented by the Callan–Symanzik equation, is a general property of Yang–Mills fields [5–10], and has been extensively studied in QCD [11–14]. The scale invariance of QCD leads to self-similar properties of strongly-interacting systems, which indeed has been observed in experimental data [15].

In a perturbative approach, self-symmetry means that any effective parton at any order is similar, after proper scaling of mass and charge, to any other effective parton at any other order of calculation. In our case, it has two important consequences: the number of degrees of freedom relevant to the process is independent of N; the probability $P(\varepsilon_0) \sim P(E)$, where $\varepsilon_o = p_o^0$ is the energy of the constituent parton and E is the energy of the initial parton, after scale is taken into account. Notice that the main parton can be, in fact, a constituent parton inside a larger system with energy $\mathcal{M}$.

The energy scale is taken into account by demanding that the ratio between the energy of the parton and the energy of its constituent partons is similar, that is:

$$\frac{\varepsilon_o}{E} \sim \frac{E}{\mathcal{M}} \sim \frac{\varepsilon}{\Lambda} \tag{13}$$

where ε is the energy of a generic constituent parton inside another parton with energy Λ. Self-similarity can be taken into account by imposing that:

$$P(\varepsilon_o/E) = P(E/\mathcal{M}) = P(\varepsilon/\Lambda). \tag{14}$$

Self-symmetry implies that a parton at any level of the fractal structure is equal to any other parton in the same structure, no matter what level of the structure it is; so far, the scaling relation is taken into account, as done above. Therefore, the probability density for the scaled parton energy must be independent of any number associated with the level it is; therefore, it cannot depend on N. In the ideal gas case, N appears in the exponent of the power-law and is related to the number of degrees of freedom that are relevant in the fractal structure. However, the probability density must be a function similar to the one obtained for the ideal gas, if the partons are to be considered at any rate as quasi-particles. Then, we suppose that the correct form for the probability density, after self-similarity is considered, is:

$$P(\varepsilon/\Lambda) = (1 - \varepsilon/\Lambda)^{\alpha}\,, \tag{15}$$

where α is a parameter to be determined, which plays the role of the number of relevant degrees of freedom to the fractal parton, as $4N - 5$ was the number of degrees of freedom for the gas of elementary particles. Furthermore, let ν be the fraction of the relevant degrees of freedom, α, that remains when we remove a part with N particles off its internal structure. Notice that, when adopting the power-law form for the probability density, we keep the general behavior of an ideal gas, but include the possibility of internal degrees of freedom for the effective partons, and therefore deviate from the ideal gas description. Then, we have from Equation (12):

$$P(\varepsilon) = \sum_n \sum_N G^n \left(\frac{N}{n(\tilde{N}-1)}\right)^4 \left(1 - \frac{\varepsilon}{\Lambda}\right)^{4N-5} \left(1 - \frac{\varepsilon}{\Lambda}\right)^{\alpha\nu}\,. \tag{16}$$

From $P(\varepsilon_0) = P(E)$, it follows immediately that:

$$\alpha = \frac{4N-5}{1-\nu}\,. \tag{17}$$

To make clear that the result obtained is equivalent to the Tsallis distribution, we define:

$$q - 1 = \frac{1}{4N-5} \tag{18}$$

and λ such that:

$$\Lambda = (4N-5)\lambda \tag{19}$$

we obtain:

$$P(\varepsilon) = \tilde{N}^r \left[1 - (q-1)\frac{\varepsilon}{\lambda}\right]^{1/(q-1)}\,, \tag{20}$$

where:

$$r = 1 - \frac{\log(q-1)}{\log \tilde{N}}\,, \tag{21}$$

$\tilde{N} = 2$ for QCD, and $q - 1$ is related to the fraction of the degrees of freedom of the whole system involved in each interaction [1]. Using scale invariance, it is possible to show that q is constant.

The main hypothesis behind Equation (10) is that for any number N, sufficiently large, of particles produced in the process, there is a very large number of possible ways to reach the final configuration such that the final result depends only on the statistical properties of the configuration achieved, loosing memory of the initial state and of the actual process that leads to the final state.

Similar results have been obtained through a different approach using the concept of thermofractals, introduced in [1] and studied in detail in [2]. There, it is shown that the fractal structure leads to the non-extensive statistics [16,17], and the relations between thermofractals and Hagedorn's self-consistent thermodynamics developed to study high energy collisions are discussed [18], and that was extended to non-extensive statistics [19].

This is obtained by increasing one order in the perturbative calculation, that is this results from a vertex function Z to which one loop is added and an external line is added to the new loop. This means that the q-exponential plays the role of an effective coupling constant in the partition function, that is:

$$Z = \mathrm{Tr}\,\langle \Psi_{n+1} | g e^{iH_o t_{n+1}} | \Psi_n \rangle \tag{22}$$

with:

$$g = \prod_{i=1}^{\tilde{N}} G \left[1 - (q-1)\frac{\varepsilon_o}{\lambda}\right]^{1/(q-1)} . \tag{23}$$

Analysis of the beta function in the one-loop approximation and comparison to the QCD result in the same approximation lead to [11–14]:

$$\frac{1}{q-1} = \frac{11}{3}c_1 - \frac{4}{3}c_2 = 7\,, \tag{24}$$

which yields $q = 1.14$. From experimental data analysis, it results $q = 1.14 \pm 0.01$, showing a good agreement between theory and experiments [20,21].

The results obtained here have shown that a system with fractal structure, similar to the thermofractals introduced in [1] and studied in [2], can be understood as a natural consequence of the scale invariance of gauge field theories. This fractal structure has been already used to investigate the properties of hadrons [22], phase-transition in hot hadronic matter [23], and neutron stars [24]. The power-law distribution of energy and momentum, which is a direct consequence of the fractal structure, was used to describe p_T distributions from high energy collision experiments [20,21,25] and to describe hadron mass spectrum [20,26]. The results obtained here give a stronger basis for the interpretation of those experimental and phenomenological studies.

The fractal structure presents at least one fractal dimension, and the Hausdorff dimension is a characteristic dimension that can be calculated by using the box-counting technique [27], where the dimension D is related to the number of boxes, $\mathcal{N}$, necessary to completely cover all possible values for the measured quantity, and D_t is the topological dimension. At some scale r, these quantities are related by [27]:

$$\mathcal{N} r^{-D} \propto r^{-D_t} . \tag{25}$$

In our case, $D_t = 1$, since we are dealing with system energy as a measure. The procedure to obtain the Hausdorff dimension is similar to that followed in [1]. The average energy of the partons at the scale λ, already introduced as the energy scale per degree of freedom, is:

$$\langle \varepsilon \rangle = \int_0^\infty \varepsilon P(\varepsilon) d\varepsilon = \frac{\lambda}{2q-1} . \tag{26}$$

The ratio between the average energy of the components and the parent system energy:

$$R = \frac{\langle \varepsilon \rangle}{E} \tag{27}$$

is related to the level of the fractal structure relative to the scale λ by:

$$R^n = \frac{\lambda}{E} = r . \tag{28}$$

The number of boxes with length λ necessary to cover the possible range of energies completely in which the fractal components can be found is $\mathcal{N} = \tilde{N}^n$. Then, it follows from Equation (28) that:

$$n = \frac{\log r}{\log R}\,. \tag{29}$$

Equation (28) also shows that, in terms of the scales, the energy of the system varies as $E \sim r^{-1}$. Let us now write the dependence of the parton energies at scale λ as $\varepsilon \sim r^{-D}$, then:

$$\mathcal{N} r^{-D} \propto r^{-1}\,, \tag{30}$$

therefore:

$$D - 1 = n \frac{\log \tilde{N}}{\log r}\,. \tag{31}$$

From Equation (29), it follows that:

$$D - 1 = \frac{\log \tilde{N}}{\log R}\,. \tag{32}$$

From Equations (26) and (27), and using $E = \lambda_r/(q-1)$, we get:

$$R = \frac{q-1}{2q-1}\,. \tag{33}$$

Using the value $q - 1 = 1.14$, it follows that $D = 0.69$. This result is in good agreement with the fractal dimension observed in intermittency analyses of high energy experimental data [28,29]. These analyses allow access to fractal dimensions by studying the behavior of cumulants of the measured distributions [30–37], and the systematics show that for pp collisions, there is a good agreement between the value obtained from the theory with those resulting from experimental data analyses.

The fact that D is fractionary is a common consequence of the fractal structure and is related to the fact that as the resolution in which the energy is measured increases, a larger number of degrees of freedom must be considered. It is important to emphasize that the fractal dimension is completely determined by the parameter q, which in Tsallis [16] statistics plays the role of the entropic index. This parameter, in turn, is completely determined by the fundamental parameters of the field theory.

In conclusion, here, we have used scaling properties to obtain the fractal structure in Yang–Mills fields and applied it to the case of QCD. non-extensivity was obtained as a consequence of the fractal structure, and the entropic index was determined in terms of the parameters of the field. The Hausdorff fractal dimension was completely determined as a function of q, and the result agrees with the values obtained in intermittency analyses of high energy collisions data.

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