Gauge Functions in Classical Mechanics: From Undriven to Driven Dynamical Systems

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Abstract: Novel gauge functions are introduced to non-relativistic classical mechanics and used to define forces. The obtained results show that the gauge functions directly affect the energy function and allow for converting an undriven physical system into a driven one. This is a novel phenomenon in dynamics that resembles the role of gauges in quantum field theories.

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1. Introduction

The background space and time of non-relativistic classical mechanics (CM) is described by the Galilean metrics $ds_1^2 = dt^2$ and $ds_2^2 = dx^2 + dy^2 + dz^2$, where t is time and x, y, and z are Cartesian coordinates that are associated with an inertial frame of reference [1]. The metrics are invariant with respect to rotations, translations and boots, which form the Galilean transformations. In Newtonian dynamics, the Galilean transformations induce a gauge transformation [2], which is called the Galilean gauge [3]. The presence of this gauge guarantees that the Newton's law of inertia is invariant with respect to the Galilean transformations, but it also shows that its Lagrangian is not [2,3].

A method to remove this gauge was recently proposed [4], and the process involves the so-called gauge functions, whose nature and origin are different than the Galilean gauge; in other words, the Galilean gauge and gauge functions are different phenomena in CM. One physical property of these functions is that they can be used to remove the unwanted Galilean gauge and make the Lagrangian Galilean invariant [4]. The main objective of this paper is to demonstrate that these gauge functions can also be used to introduce forces into otherwise undriven dynamical systems.

Different gauge transformations are known in CM and they lead to infinite gauge potentials, which in the zero-order become the electromagnetic potentials, and in the first-order are identified as the electromagnetic and gravitational potentials [5,6]. Gauge transformations in the Lagrangian and Hamiltonian formalism of CM, and the resulting diffeomorphism-induced gauge symmetries in CM, were also investigated [7], with applications to General Relativity. However, these gauge transformations and their studies are not relevant to the gauge functions that are described in this paper.

In this paper, we generalize the gauge functions derived in [4], and use them to account for external forces acting on a dynamical system. We present a general method to find these gauge functions and apply them to simple (linear, undamped, undriven, and one-dimensional) oscillators, with the purpose to demonstrate how such undriven oscillators can be converted into driven ones. It is suggested that the presented method can be applied to other dynamical systems and these gauge function-introduced forces may give more physical insight into the connection between forces in CM and gauge-introduced interactions in quantum field theory (QFT) [8].

For the simple oscillators, the independent variable t is time and the dependent variable $x(t)$ is a displacement. Let $\hat{D} = d^2/dt^2 + c$ be an linear differential operator, with c being a constant whose

value may change from one dynamical system to another, and let $\mathcal{Q}$ be a set of all ordinary differential equations (ODEs) of the form $\hat{D}x(t) = 0$; depending on the physical meaning of $x(t)$ and c, the ODEs of $\mathcal{Q}$ may describe different oscillators, including pendulums. General solutions of these ODEs are well-known and they can be written as $x(t) = c_1 x_1(t) + c_2 x_2(t)$, where c_1 and c_2 are integration constants, and $x_1(t)$ and $x_2(t)$ are the solutions given in terms of the elementary functions [9,10].

The Lagrangian formalism is established for the ODEs of $\mathcal{Q}$. The formalism has always played an important role in obtaining equations of motion of dynamical systems [10]. For the conservative dynamical systems, the existence of Lagrangians is guaranteed by the Helmholtz conditions [11], which can also be used to derive the Lagrangians. The procedure of finding the Lagrangians is called the inverse (or Helmholtz) problem of calculus of variations and there are different methods to solve this problem [12,13]. We solve the Helmholtz problem and find two families of Lagrangians that are classified as primary and general. Within each family, two separate classes of Lagrangians are considered, namely, standard and null Lagrangians.

For standard Lagrangians (SLs), the kinetic and potential energy, like terms and the term with the square of dependent variable, are easily identified [10,12,13], and these Lagrangians have been known since the original work of Lagrange in the 18th century. On the other hand, null (or trivial) Lagrangians (NLs) contain neither the kinetic nor potential energy like terms, and they make the Euler–Lagrange (E–L) equation to vanish identically. Moreover, NLs can also be expressed as the total derivative of a scalar function [14,15], which is called a gauge function [3]. Our main objective is to obtain the gauge functions for the constructed NLs for the ODEs of $\mathcal{Q}$.

The fact that the NLs and their gauge functions can be omitted when the original equations are derived is obvious (e.g., [2,3]); however, it is also commonly recognized that the NLs are important in studies of symmetries of Carathéodory's theory of fields of extremals and in integral invariants [15,16]. There is a large body of literature on the NLs and on their mathematical applications (e.g., [17–21]). Moreover, the NLs play an important role in studies of elasticity, where they physically represent the energy density function of materials [22,23], and making Lagrangians invariant in the Galilean invariant theories [4].

The main goals of this paper are: (i) construction of the SLs and NLs and the gauge functions corresponding to the NLs; (ii) using these gauge functions to determine the energy function and define forces; (iii) deriving new SLs that give the equation of motion with the forces; (iv) identifying the gauge functions that can be used to define forces in CM; and, (v) using the gauge functions to convert an undriven oscillator into a driven one. The presented approach is self-consistent and it shows that introducing the gauge functions into CM is the equivalent of defining the time-dependent driving forces.

The outline of the paper is as follows: in Section 2, the Principle of Least Action and Lagrangians are described; Section 3 deals with the Lagrangian formalism for the considered ODEs and the gauge functions are also derived; in Section 4, the energy function for the gauge functions, new definition of forces, and the resulting inhomogeneous equations of motion for oscillators with different forces are presented and discussed; finally, Section 5 provides our conclusions.

2. Principle of Least Action and Lagrangians

The Lagrange formalism deals with a functional $\mathcal{A}[x(t)]$, where $\mathcal{A}$ is the action and $x(t)$ is an ordinary and smooth function to be determined. Typically, $\mathcal{A}[x(t)]$ is given by an integral over a smooth function $L(\dot{x}, x, t)$, which is called a Lagrangian and $\dot{x}$ is a derivative of x with respect of t. The integral defined in this way is mathematical representation of the Principle of Least Action or Hamilton's Principle [24], which requires that $\delta\mathcal{A} = 0$, where δ is the variation known also as the functional (Fréchet) derivative of $\mathcal{A}[x(t)]$ with respect to $x(t)$. Using $\delta\mathcal{A} = 0$, the E–L equation is obtained, and this equation is a necessary condition for the action to be stationary (to have either a minimum or maximum or saddle point).

We solve the inverse problem of the calculus of variations for the ODEs of Q and find their SLs and NLs; the validity of the Helmholtz conditions [6] for these Lagrangians is also discussed. Different methods were previously developed to determine the SLs for different ODEs [25–34] and some of these methods [25,26] will be used in the next section. Based on the original work of Lagrange in the 18th century, the SLs contain the difference between the kinetic and potential energy like terms, which, here, will be represented by the difference between the square of the first order derivative of the dependent variable and the term with the square of dependent variable [10,12,13].

On the other hand, the NLs contain neither kinetic nor potential energy like terms but instead they depend on terms with a mixed dependent variable and its derivative [14,20], *and terms with the mixed dependent variable (or its derivative) with the independent variable, and also terms that only depend on the dependent variable*. The derived NLs are new and they are restricted to the lowest order in the dependent variable. For any NL, the E–L equation identically vanishes, and any NL can be expressed as the total derivative of their gauge functions. Our main results are novel gauge functions obtained for the NLs and their role in converting undriven dynamical systems into driven ones.

3. Lagrangians and Gauge Functions

3.1. Standard and Null Lagrangians

Using the definition of SLs given in Section 2, they can be written in the following form

$$L_s[\dot{x}(t), x(t)] = \frac{1}{2} \left[\alpha \left(\dot{x}(t) \right)^2 + \beta x^2(t) \right] , \tag{1}$$

where the coefficients α and β are either constants or functions of time; most SLs obtained here are already known [10,12,13]. The NLs are defined in Section 2, and let us point out that the derived NLs are new. We now follow [4] to show how NLs with constant coefficients can be constructed and then generalize this approach in Section 3.3.

Let $L_m[\dot{x}(t), x(t)]$ be a mixed Lagrangian of the dependent and independent variables given by

$$L_m[\dot{x}(t), x(t), t] = C_1\dot{x}(t)x(t) + C_2\dot{x}(t)t + C_3x(t)t , \tag{2}$$

and $L_f[\dot{x}(t), x(t)]$ be a Lagrangian of the single dependent variable, written as

$$L_f[\dot{x}(t), x(t)] = C_4\dot{x}(t) + C_5x(t) + C_6 , \tag{3}$$

where C_1, C_2, C_3, C_4, C_5, and C_6 are arbitrary constants. However, with $x(t)$ being a displacement of harmonic scillators and t being time, the constants must have different physical dimensions to get the same dimensions of $L_m[\dot{x}(t), x(t), t]$ and $L_f[\dot{x}(t), x(t)]$, as that of $L_s[\dot{x}(t), x(t)]$.

We define $\hat{EL}$ to be the E–L equation operator and take $\hat{EL}(L_m + L_f) = 0$, which is required for $L_n[\dot{x}(t), x(t), t] = L_m[\dot{x}(t), x(t), t] + L_f[\dot{x}(t), x(t)]$ to become the null Lagrangian. This is true if, and only if, $C_3 = 0$ and $C_5 = C_2$. Subsequently, the null Lagrangian can be written [4] as

$$L_n[\dot{x}(t), x(t), t] = \sum_{i=1}^{4} L_{ni}[\dot{x}(t), x(t), t] , \tag{4}$$

where $i = 1, 2, 3,$ and 4, and the partial NLs are given by $L_{n1}[\dot{x}(t), x(t)] = C_1\dot{x}(t)x(t)$, $L_{n2}[\dot{x}(t), x(t), t] = C_2[\dot{x}(t)t + x(t)]$, $L_{n3}[\dot{x}(t)] = C_4\dot{x}(t)$ and $L_{n4} = C_6$; with $L_{n2}[\dot{x}(t), x(t), t]$ being the only partial null Lagrangian that depends explicitly on t. Note that these partial null Lagrangians are constructed to lowest orders of the dynamic variable $x(t)$.

Since $L_n[\dot{x}(t), x(t), t] = d\Phi_p/dt$, we may write the gauge function $\Phi_p(t)$ [4] as

$$\Phi_p(t) = \sum_{i=1}^{4} \phi_{pi}(t) \, , \tag{5}$$

where the partial gauge functions $\phi_{pi}(t)$ correspond the partial null Lagrangians $L_{ni}[\dot{x}(t), x(t)]$, and they are defined as $\phi_{p1}(t) = C_1 x^2(t)/2$, $\phi_{p2}(t) = C_2 x(t)t$, $\phi_{p3}(t) = C_4 x(t)$ and $\phi_{p4}(t) = C_6 t$.

We now use the above results to derive the SLs, NLs, and gauge functions for the ODEs of $\mathcal{Q}$.

3.2. Primary Lagrangians and Gauge Functions

We consider the ODEs of $\mathcal{Q}$ and write them in their explicit form

$$\ddot{x}(t) + cx(t) = 0 \, , \tag{6}$$

where c may be any real number. Let us define the following primary Lagrangian

$$L_p[\dot{x}(t), x(t), t] = L_{ps}[\dot{x}(t), x(t)] + L_{pn}[\dot{x}(t), x(t), t] \, , \tag{7}$$

where the primary standard Lagrangian (with $\alpha = 1$ and $\beta = -c$ in Equation (1)) is given by

$$L_{ps}[\dot{x}(t), x(t)] = \frac{1}{2} \left[(\dot{x}(t))^2 - cx^2(t) \right] \, , \tag{8}$$

and the primary null Lagrangian $L_{pn}[\dot{x}(t), x(t)]$ is equal to $L_n[\dot{x}(t), x(t)]$ (see Equation (4)) with the same partial NLs. In addition, the primary gauge function $\Phi_p(t)$ is given by Equation (5) with the same partial gauge functions.

3.3. General Lagrangians and Gauge Functions

The above results can be generalized by writing the Lagrangian given by Equation (1) in the following form

$$L_s[\dot{x}(t), x(t)] = \frac{1}{2} \left[\alpha(t) (\dot{x}(t))^2 + \beta(t)x^2(t) \right] \, , \tag{9}$$

where $\alpha(t)$ and $\beta(t)$ are continuous and differentiable functions. Substituting this Lagrangian to the E-L equation, we find $\alpha(t) = C_0$ and $\beta(t) = -C_0 c$, where C_0 is an intergration constant. Subsequently, the general standard Lagrangian can be written as

$$L_{gs}[\dot{x}(t), x(t)] = \frac{1}{2}C_0 \left[(\dot{x}(t))^2 - cx^2(t)) \right] \, . \tag{10}$$

This Lagrangian can be reduced to the primary standard Lagrangian if $C_0 = 1$ and it can also be used to define the following general Lagrangian

$$L_g[\dot{x}(t), x(t), t] = L_{gs}[\dot{x}(t), x(t), t] + L_{gn}[\dot{x}(t), x(t), t] \, , \tag{11}$$

where the general null Lagrangian is

$$L_{gn}[\dot{x}(t), x(t), t] = \sum_{i=1}^{4} L_{gni}[\dot{x}(t), x(t), t] \, , \tag{12}$$

with $\hat{EL}(L_{gn}) = 0$ and $L_{gni}[\dot{x}(t), x(t), t]$ being its partial components. To determine the partial null Lagrangians, we generalize the primary gauge functions $\phi_{pi}(t)$ given below Equation (5) by replacing

their constant coefficients by functions of the independent variable t. Denoting the general gauge functions as $\phi_{gi}(t)$, we obtain

$$\phi_{g1}(t) = \frac{1}{2}f_1(t)x^2(t), \tag{13}$$

$$\phi_{g2}(t) = f_2(t)x(t)t, \tag{14}$$

$$\phi_{g3}(t) = f_4(t)x(t), \tag{15}$$

and

$$\phi_{g4}(t) = f_6(t)t, \tag{16}$$

where $f_1(t)$, $f_2(t)$, $f_4(t)$, and $f_6(t)$ are continuous and differentiable functions to be determined.

Subsequently, we take the total derivative of these partial gauge functions and obtain the following partial Lagrangians

$$L_{gn1}[\dot{x}(t), x(t), t] = \left[f_1(t)\dot{x}(t) + \frac{1}{2}\dot{f}_1(t)x(t) \right] x(t), \tag{17}$$

$$L_{gn2}[\dot{x}(t), x(t), t] = \left[\left(f_2(t)\dot{x}(t) + \dot{f}_2(t)x(t) \right) t + f_2(t)x \right], \tag{18}$$

$$L_{gn3}[\dot{x}(t), x(t), t] = \left[f_4(t)\dot{x}(t) + \dot{f}_4(t)x(t) \right], \tag{19}$$

and

$$L_{gn4}[\dot{x}(t), x(t), t] = \left[\dot{f}_6(t)t + f_6(t) \right], \tag{20}$$

which can be added together to obtain the general null Lagrangian (see Equation (12)). This Lagrangian depends on four functions that must be continous and differentiable, but otherwise arbitrary. The specification of initial conditions for physical problems would set up constraints on these functions; however, in this paper the functions are kept arbitrary for reasons explained in Section 4.

The general null Lagrangian reduces to the primary null Lagrangian when $f_1(t) = C_1$, $f_2(t) = C_2$, $f_4(t) = C_4$ and $f_6(t) = C_6$.

3.4. Discussion of Lagrangians and Gauge Functions

The obtained results show that the SLs and NLs can be found for the ODEs of $\mathcal{Q}$ by solving the inverse problem of the calculus of variations. The existence of these Lagrangians must be validated by the Helmholtz conditions [11]. There are three original Helmholtz conditions and it is easy to verify that all of the Lagrangians constructed for $\hat{D}x(t) = 0$ obey these conditions, which means that the SLs do exist for undamped (conservative) systems [5–8]. Let us also point out that the existence of NLs is not affected by the Helmholtz conditions, because these Lagrangians have no effects on the derivation of the original equations.

We derived the primary and general SLs and NLs for the ODEs of $\mathcal{Q}$. Most of the obtained SLs are already known and they are generated as a byproduct of our procedure of deriving the NLs, which are new for the considered equations. For each null Lagrangian, we found its corresponding gauge function. The general Lagrangians depend on four functions that must be continuous and differentiable, and must satisfy initial conditions of a specific physical problem. If the functions are assumed to be constants, the primary NLs are obtained. Because the functions are arbitrary, many different NLs can be obtained by choosing different forms of these functions.

It was previously demonstrated that a Lagrangian is null if, and only if, it can be represented as the total derivative of a scalar function of the system variables [14]. If this function exists, the resulting transformation is called the gauge transformation and the function is known as a gauge function [3,4]. The results presented here demonstrate that this definition is valid for the ODEs of $\mathcal{Q}$, and that for all of these equations the gauge functions exist. The presented gauge functions were derived here only for the ODEs with the constant coefficients; notably, the derivations can also be extended to the ODEs with non-constant coefficients and first attempts in finding such gauge functions are described in [33,34].

Because the obtained NLs are given as total derivatives of scalar functions, they can be omitted from the Lagrangians when the original equations are derived from the E–L equation [2,3,35]. However, the purpose of this paper is to determine the NLs and derive the corresponding gauge functions, which are then used to convert undriven for oscillators (or pendulums) into driven systems. In other words, the gauge functions are used to introduce forces to CM, which is a new phenomenon.

4. Application: From Undriven to Driven Oscillators

4.1. Primary Gauge and Energy Functions

Let us consider a harmonic oscillator and identify $x(t)$ with its displacement variable. The equation of motion of the oscillator is $\hat{D}x(t) = 0$ with $c = k/m$, where k is a spring constant and m is mass. The characteristic frequency of the oscillator is then $\omega_0 = \sqrt{c} = \sqrt{k/m}$, and the equation of motion can be written as

$$\ddot{x}(t) + \omega_0^2 x(t) = 0 . \tag{21}$$

It must be noted that Equation (21) also describes a linear and undamped pendulum if $x(t)$ is replaced by $\theta(t)$, where $\theta(t)$ is an angle of the pendulum, and ω_0 is replaced by the pendulum characteristic frequency $\omega_p = \sqrt{c} = \sqrt{g/L}$, where g is gravitational acceleration and L is length of the pendulum. With these replacements, the results presented below for the oscillator are also valid for the pendulum.

According to Equation (7), the primary Lagrangian $L_p[\dot{x}(t), x(t)]$ for these harmonic oscillators can be written as

$$L_p[\dot{x}(t), x(t)] = L_{ps}[\dot{x}(t), x(t)] + \frac{d\phi_p}{dt} , \tag{22}$$

where the primary standard Lagrangian is given by

$$L_{ps}[\dot{x}(t), x(t)] = \frac{1}{2} \left[(\dot{x}(t))^2 - \omega_0^2 x^2(t) \right] , \tag{23}$$

and the primary gauge function Φ_p is

$$\Phi_p(t) = \sum_{i=1}^{4} \phi_{pi}(t) , \tag{24}$$

and the partial primary gauge functions are:

$$\phi_{p1} = \frac{1}{2} C_1 x^2(t) , \tag{25}$$

$$\phi_{p2} = C_2 x(t) t , \tag{26}$$

$$\phi_{p3} = C_4 x(t) , \tag{27}$$

and

$$\phi_{p4} = C_6 t . \tag{28}$$

Note that the total derivative of each one of these partial gauge functions gives no contribution to the resulting equation of motion. However, these gauge functions may be used to impose Galilean invariance of SLs [4].

Because the gauge functions ϕ_{p2} and ϕ_{p4} explicitly depend on time t, the resulting primary null Lagrangian is also a function of time. This requires that the primary energy function, E_p, is calculated [36,37] using

$$E_p[\dot{x}(t), x(t)] = \dot{x} \frac{\partial L_p}{\partial \dot{x}} - L_p[\dot{x}(t), x(t)] , \tag{29}$$

which gives

$$E_{\mathrm{p}}[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 + \omega_0^2 x^2(t)\right] - [C_2 x + C_6] \, , \tag{30}$$

with the first two terms on the RHS representing the energy function E_{ps} for the primary standard Lagrangian and the other two terms corresponding to the primary energy function E_{pf} for the primary gauge function, so that $E_{\mathrm{p}} = E_{\mathrm{ps}} + E_{\mathrm{pf}}$.

In general, $E_{\mathrm{p}} \neq E_{\mathrm{tot}}$, with $E_{\mathrm{tot}} = E_{\mathrm{ps}} = H_{\mathrm{ps}}$, where E_{tot} is the total energy of system and H_{ps} is its Hamiltonian, corresponding to the primary standard Lagrangian, and given by $H_{\mathrm{ps}} = E_{\mathrm{p}} - E_{\mathrm{pf}}$ or

$$H_{\mathrm{ps}}[\dot{x}(t), x(t)] = \frac{1}{2}\left[\dot{x}^2(t) + \omega_0^2 x^2(t)\right] \, . \tag{31}$$

Using the Hamilton equations, the equation of motion for the harmonic oscillator given by Equation (21) is obtained. A similar result is derived when the total derivative of E_{p} is equal to the negative partial time derivative of L_{p}, which can be written [36] as

$$\frac{dE_{\mathrm{p}}}{dt} = -\frac{\partial L_{\mathrm{p}}}{\partial t} \, , \tag{32}$$

which again gives Equation (21). It must be noted that E_{p} is a conserved quantity and that $E_{\mathrm{p}} \neq E_{\mathrm{tot}}$. This shows that the equation of motion of the harmonic oscillator is also obtained when the energy function is used instead of the primary Lagrangian L_{p} or the Hamiltonian H_{ps}.

The above results show that among the four primary gauge functions, $\phi_{\mathrm{p}1}$, $\phi_{\mathrm{p}2}$, $\phi_{\mathrm{p}3}$, and $\phi_{\mathrm{p}4}$, the first and third do not contribute to the primary energy function, but the second and fourth do contribute although each one differently. The partial gauge function $\phi_{\mathrm{p}2}$ breaks into two parts and only the part that depends on $C_2 x$ contributes to the energy function. However, the partial gauge function $\phi_{\mathrm{p}4}$ fully contributes to the energy function. Let us call $\phi_{\mathrm{p}2}$ the primary F-gauge function, and $\phi_{\mathrm{p}4}$ the primary E-gauge function.

The reasons for these names follows. First, the term $C_2 x$ represents energy if, and only if, the coefficient C_2 is a constant acceleration, or a constant force per mass, so that $C_2 x$ is work done by this force on the system. This clearly shows that the primary partial gauge function $\phi_{\mathrm{p}2}$ can be used to introduce forces that cause the constant acceleration. Second, the primary partial gauge function $\phi_{\mathrm{p}4}$ introduces a constant energy shift in the system.

Let us define $F_{\mathrm{c}} = C_2$, where F_{c} represents a constant acceleration or constant force per mass. Similarly, $E_{\mathrm{c}} = C_6$ is a constant energy shift that could be caused by the force. Subsequently, the primary energy function can be written as

$$E_{\mathrm{p}}[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 + \omega_0^2 x^2(t)\right] - [F_{\mathrm{c}} x + E_{\mathrm{c}}] \, . \tag{33}$$

This demonstrates that some gauge functions can be used to introduce external forces that drive the system but other gauge functions may either generate a shift of the total energy of the system, or simply have no effect on the system. In other words, only gauge functions that depend explicitly on time may be used to introduce forces in CM. These are new phenomena caused exclusively by including the gauge functions into CM.

4.2. General Gauge and Energy Functions

The above results can be now extended to the general standard and null Lagrangians and their gauge functions with application to a harmonic oscillator and pendulum. According to Equations (10) through (12), the general Lagrangian for the oscillator can be written as

$$L_{\mathrm{g}}[\dot{x}(t), x(t), t] = L_{\mathrm{gs}}[\dot{x}(t), x(t), t] + L_{\mathrm{gn}}[\dot{x}(t), x(t), t] \, , \tag{34}$$

where the general standard and null Lagrangians are

$$L_{gs}[\dot{x}(t), x(t)] = \frac{1}{2}C_0 \left[(\dot{x}(t))^2 - \omega_0^2 x^2(t) \right] \,, \tag{35}$$

and

$$L_{gn}[\dot{x}(t), x(t), t] = \sum_{i=1}^{4} \frac{d\phi_{gi}}{dt} \,, \tag{36}$$

with the partial gauge functions $\phi_{gi}(t)$ being given by Equations (13) through (16).

The general energy function, $E_g[\dot{x}(t), x(t)]$, can be calculated by substituting $L_{gn}[\dot{x}(t), x(t), t]$ into Equation (29), which gives

$$E_g[\dot{x}(t), x(t)] = E_{gs}[\dot{x}(t), x(t)] + E_{gf}[\dot{x}(t), x(t)] \,, \tag{37}$$

where the general energy function for the general standard Lagrangian is

$$E_{gs}[\dot{x}(t), x(t)] = \frac{1}{2}C_0 \left[(\dot{x}(t))^2 + \omega_0^2 x^2(t) \right] \,. \tag{38}$$

and the general energy function for the general gauge function can be written as

$$E_{gf}[\dot{x}(t), x(t)] = - \left[\frac{1}{2}\dot{f}_1(t)x^2(t) + \dot{f}_2(t)x(t)t \right]$$

$$- \left[\left(f_2(t) + \dot{f}_4(t) \right) x(t) + f_6(t) + \dot{f}_6(t)t \right] \,. \tag{39}$$

Because $E_{gs} = H_{gs} = E_{tot}$, then $H_{gs} = E_g - E_{gf}$ and, as expected, when H_{gs} is substituted into the Hamilton equations, the equation of motion for the harmonic oscillator (see Equation (21)) is obtained. The same equation of motion is derived when the total derivative of E_g is equal to the negative partial time derivative of L_g (see Equation (32)).

The obtained results show that the general gauge functions, ϕ_{g1} and ϕ_{g3} also contribute to the general energy function, in addition, to the ϕ_{g2} and ϕ_{g4} contributions. We generalize the previous definitions and now call ϕ_{g2} the *general F-gauge function*, and ϕ_{g4} the general E-gauge function. However, no special names are given to the gauge functions ϕ_{g1} and ϕ_{g3}, and only their contributions to forces is shown below.

We may define the following functions: $F(t, x) = [f_2(t) + \dot{f}_2(t)t + \dot{f}_4(t)]x(t)$ and $G(t) = f_6(t) + \dot{f}_6(t)t$ and see that all gauge functions contribute to them. Using these definitions, we write

$$E_g[\dot{x}(t), x(t)] = \frac{1}{2} \left[(\dot{x}(t))^2 + \omega_0^2 \left(1 - \frac{\dot{f}_1(t)}{\omega_0^2} \right) x^2(t) \right]$$

$$- [F(t, x) + G(t)] \,, \tag{40}$$

which shows that the gauge functions allow us to introduce two functions, one that depends linearly on displacement, but is arbitrary in time, and the other that is an arbitrary function of time only. This general formula for the energy function may be further simplified by taking $f_1(t) = C_1 = $ const, which means that the shift of the potential energy is not time-dependent and remains constant all the time. Subsequently, the general energy function becomes

$$E_g[\dot{x}(t), x(t)] = \frac{1}{2} \left[(\dot{x}(t))^2 + \omega_0^2 x^2(t) \right] - [F(t, x) + G(t)] \,, \tag{41}$$

and the function $F(t, x)$ reduces to the primary energy function if $F(t, x) = F_c x(t)$, and the function $G(t)$ becomes E_c (see Equation (33)).

4.3. Time-Dependent Forces

Having obtained the general energy function $E_g[\dot{x}(t), x(t)]$, for the equations of motion of undriven oscillators, we now demonstrate that these systems can be converted into driven ones. This can be done by adding the extra terms $F(t, x)$ and $G(t)$ to the general standard Lagrangian. Let us separate the dependent and independent variables in $F(t, x)$ and write $F(t, x) = \mathcal{F}(t)x(t)$. The result is

$$L_g[\dot{x}(t), x(t)] = L_{gs}[\dot{x}(t), x(t)] + [\mathcal{F}(t)x + G(t)] , \tag{42}$$

where $L_{ps}[\dot{x}(t), x(t)]$ and $L_{gs}[\dot{x}(t), x(t)]$ are given by Equations (23) and (35), respectively. Substituting $L_g[\dot{x}(t), x(t)]$ into the E–L equation, we obtain

$$\ddot{x}(t) + \omega_0^2 x(t) = \mathcal{F}(t) . \tag{43}$$

This equation describes a driven oscillator with $\mathcal{F}(t)$ being a time-dependent force. The equation also represents a linear undamped pendulum if $x(t)$ is replaced by $\theta(t)$ and ω_0 is replaced by ω_p. In a special case of the primary null Lagrangian with constant coefficients (see Equation (7)), the force $\mathcal{F}(t)$ is the constant force F_c.

Let us point out that the resulting inhomogeneous equation of motion is also obtained from the Hamilton equations when the energy function is used instead of the Hamiltonians $H_{ps}[\dot{x}(t), x(t)]$ and $H_{gs}[\dot{x}(t), x(t)]$. This is expected as the Hamiltonians represent the total energy of the system, which is not conserved, but the energy function is a constant of motion for the considered driven harmonic oscillator. This shows that our approach is self-consistent and based on the principles of CM. However, by accounting for the gauge functions and by showing their relationships to forces, this paper describes a new phenomenon in CM, which formally allows converting undriven dynamical systems into driven ones. The converting process can be used for any linear dynamical system for whose equation of motion is known.

5. Conclusions

The Lagrangian formalism was established for equations describing different undriven dynamical systems by constructing the standard and null Lagrangians, and the gauge functions corresponding to the latter. The gauge functions were used to determine the energy function and define forces. Using these forces, new standard Lagrangians were obtained and the equations of motion resulting from these Lagrangians were derived. It was shown that the equations of motion are inhomogeneous because of the presence of the time-dependent driving forces introduced by the gauge functions, and that the same equations can be obtained by using either the energy function or the Hamilton equations. Moreover, the obtained results demonstrate that the approach does not allow defining dissipative forces that depend on velocities. It was also pointed out that only some gauge functions give the driving forces and those gauge functions were identified and discussed.

The presented approach is self-consistent and it shows that introducing the gauge functions into classical mechanics is equivalent of finding the time-dependent driving forces; it must be noted that the gauge functions derived in this paper are different from the gauges considered before. The obtained results demonstrate that not all gauge functions give forces; instead, there is only one primary and only one general gauge function that introduces the driving forces to classical mechanics. This new phenomenon of defining the driving forces in classical mechanics by the gauge functions, and converting an undriven system into a driven one, can be easily generalized to other linear dynamical systems, either conservative or non-conservative. The phenomenon resembles the role of gauges in quantum field theories but there are differences in the underlying physics that will be investigated separately.

References

1.	Arnold, V.I. *Mathematical Methods of Classical Mechanics*; Springer: New York, NY, USA; Heidelberg/Berlin, Germany, 1978.

2.	Landau, L.D.; Lifschitz, E.M. *Mechanics*; Pergamon Press: Oxford, UK, 1969.

3.	Levy-Leblond, J.-M. Group theoretical foundations of classical mechanics: Teh Lagrange gauge problem. *Commun. Math. Phys.* **1969**, *12*, 64. [CrossRef]

4.	Musielak, Z.E.; Watson, T.B. Gauge functions and Galilean invariance of action. *Phys. Let. A* **2020**, *384*, 126642. [CrossRef]

5.	Boisseau, B.; Barrbes, C. Gauge theory in Hamiltonian classical mechanics. *J. Math. Phys.* **1979**, *20*, 2058. [CrossRef]

6.	Filippov, V.M.; Savchin, V.M.; Shorokhov, S.G. Variational principles for nonpotential operators. *J. Math. Sci.* **1994**, *68*, 275. [CrossRef]

7.	Pons, J.M.; Salisbury, D.C.; Shepley, L.C. Gauge transformations in the Lagrangian and Hamiltonian formalisms of generally covariant theories. *Phys. Rev. D* **1997**, *55*, 658. [CrossRef]

8.	Frampton, P.H. *Gauge Field Theories*; John Wiley & Sons, Inc.: New York, NY, USA, 2000.

9.	Murphy, G.M. *Ordinary Differential Equations and Their Solutions*; Dover Publication, Inc.: New York, NY, USA, 2011.

10.	Teschl, G. *Ordinary Differential Equations and Dynamical Systems*; American Mathematical Society Press: Providence, RI, USA, 2012.

11.	von Helmholtz, H. On physical meaning of the principle of least action. *J. Reine Angew. Math.* **1887**, *100*, 213.

12.	Lopuszanski, J. *The Inverse Variational Problems in Mechanics*; World Scientific: Singapore, 1999.

13.	Vujanovic, B.D.; Jones, S.E. (Eds.) *Variational Methods in Nonconservative Phenomena*; Academic Press: New York, NY, USA, 1989; Volume 182, pp. 1–371.

14.	Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1993.

15.	Betounes, D.E. Differential geometric aspects of the Cartan form: Symmetry theory. *J. Math. Phys.* **1987**, *28*, 2347. [CrossRef]

16.	Olver, P.J.; Sivaloganathan, J. The structure of null Lagrangians. *Nonlinearity* **1989**, *1*, 389. [CrossRef]

17.	Grigore, D.R. Trivial second-order Lagrangians in classical field theory. *J. Phys. A* **1995**, *28*, 2921. [CrossRef]

18.	Krupka, D.; Musilova, J. Trivial Lagrangians in field theory. *Diff. Geom. Its Appl.* **1998**, *9*, 225. [CrossRef]

19.	Vitolo, R. On different geometric formulations of Lagrange formalism. *Diff. Geom. Its Appl.* **1999**, *10*, 293.

20.	Crampin, M.; Saunders, D.J. On null Lagrangians. *Diff. Geom. Its Appl.* **2005**, *22*, 131. [CrossRef]

21.	Krupkova, D.K.O.; Saunders, D. The Cartan form and its generalizations in the calculus of variations. *Int. J. Geom. Meth. Mod. Phys.* **2010**, *7*, 631. [CrossRef]

22.	Anderson, D.R.; Carlson, D.E.; Fried, E. A continuum-mechanical theory for nematic elastomers. *J. Elast.* **1999**, *56*, 35. [CrossRef]

23.	Saccomandi, G.; Vitolo, R. Null Lagrangians for nematic elastomers. *J. Math. Sci.* **2006**, *136*, 4470. [CrossRef]

24.	Doughty, N.A. *Lagrangian Interaction*; Addison-Wesley: New York, NY, USA, 1990.

25.	Musielak, Z.E. Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients. *J. Phys. A Math. Theor.* **2008**, *41*, 055205. [CrossRef]

26.	Cieśliński, J.L.; Nikiciuk, T. A direct approach to the construction of standard and non-standard Lagrangians for dissipative-like dynamical systems with variable coefficients. *J. Phys. A Math. Theor.* **2008**, *43*, 175205. [CrossRef]

27. Sarlet, W. On equivalent Lagrangians and symmetries. *J. Phys. A* **1982**, *1*, 1203.

28. Nucci, M.C.; Leach, P.G.L. Leach, Lagrangian galore. *J. Math. Phys.* **2007**, *48*, 123510. [CrossRef]

29. Choudhury, A.G.; Guha, P.; Khanra, B. On the Jacobi last multiplier, intergrating factors and the Lagrangian formulation of differential equations of the Painlevé-Gambier classification. *J. Math. Anal. Appl.* **2009**, *360*, 651. [CrossRef]

30. Riewe, F. Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E* **1997**, *55*, 3581. [CrossRef]

31. Costa, D.G. *An Invitation to Variational Problems in Differential Equations*; Birkhäuser: Boston, MA, USA, 2007.

32. Davachi, N.; Musielak, Z.E. Generalized non-standard Lagrangians. *J. Undergrad. Rep. Phys.* **2019**, *29*, 100004. [CrossRef]

33. Musielak, Z.E.; Davachi, N.; Rosario-Franco, M. Special functions of mathematical physics: A unified Lagrangian formalism. *Mathematics* **2020**, *8*, 389.

34. Musielak, Z.E.; Davachi, N.; Rosario-Franco, M. Rosario-Franco, Lagrangians, gauge functions, and Lie groups for semigroup of second-order differential equations. *J. Appl. Math.* **2020**, *2020*, 3170130. [CrossRef]

35. Basdevant, J.-L. *Variational Principles in Physics*; Springer: New York, NY, USA, 2007.

36. Goldstein, H.; Poole, C.P.; Safko, J.L. *Classical Mechanics*, 3rd ed.; Addison-Wesley: San Francisco, CA, USA, 2002.

37. José, J.V.; Saletan, E.J. *Classical Dynamics, A Contemporary Approach*; Cambridge University Press: Cambridge, UK, 2002.